Math 3140 — Fall 2012
Handout #10

Exercise 1. (a) Show that $\mathbf{Z}/2\mathbf{Z}$ is a field.

(b) Show that $\mathbf{Z}/3\mathbf{Z}$ is a field.

(c) Find an element of $\mathbf{Z}/4\mathbf{Z}$ that doesn't have a multiplicative inverse.

(d) Is $\mathbf{Z}/5\mathbf{Z}$ a field?

(e) Is $\mathbf{Z}/6\mathbf{Z}$ a field?

(f) Is $\mathbf{Z}/7\mathbf{Z}$ a field?

(g) Formulate a conjecture about when $\mathbf{Z}/n\mathbf{Z}$ is a field.

Exercise 2. (a) Recall that $\mathbf{C}$ is the ring consisting of all symbols $a + bi$ where $a, b \in \mathbf{R}$. The addition and multiplication rules are

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Find all solutions to the equation $x^2 = 0$ in the ring $\mathbf{C}$.

(b) Let A be the ring consisting of all symbols $a + b\epsilon$ where $a, b \in \mathbf{R}$. Define

$$(a + b\epsilon) + (c + d\epsilon) = (a + c) + (b + d)\epsilon$$
$$(a + b\epsilon)(c + d\epsilon) = ac + (ad + bc)\epsilon.$$

Find all solutions to the equation

$$x^2 = 0$$

in A.

Exercise 3. Let M be the ring of 2×2 matrices with real entries.

(a) Is M a division algebra?

(b) Let $R \subset M$ be the subset of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with $a, b \in \mathbf{R}$. Convince yourself that R is a ring (check that addition, subtraction, and multiplication are of elements of R are all in R). This is an example of a **subring**.

(c) Is R commutative?

(d) Is R a field?

(e) We haven't defined isomorphisms of rings yet, but intuitively rings are isomorphic if there is a bijection between making their ring structures compatible. Is R isomorphic to a familiar ring?

Exercise 4. Let R be the set of symbols $a + bi$ where $a, b \in \mathbf{Z}$. Define addition and multiplication of elements of R using addition and multiplication of complex numbers.

(a) Convince yourself that R is a ring. (Check that addition, subtraction, and multiplicaiton are all well-defined.)

An element x of a ring A is said to be **irreducible** if whenever $x = yz$ for elements $y, z \in A$ either y or z must have a multiplicative inverse. Note that the irreducible elements of **Z** are the prime numbers. An element of a ring that is not irreducible is called **reducible**.

(b) Show that the number element $2 = 2 + 0i$ in R is reducible. (Show, in other words, that 2 can be factored into a product of elements of R that are not invertible.)

(c) Show that the number $5 = 5 + 0i$ in R is reducible.

(d) Is 3 reducible?

(e) Which other primes from **Z** are reducible? Which are irreducible?