Math 3140 — Fall 2012
Handout #6

Exercise 1. Let $U = \{z \in \mathbf{C} \mid |z| = 1\}$ be the group of complex numbers of absolute value 1. For each $z \in U$, let

$$T_z : \mathbf{C} \to \mathbf{C}$$

be the function $T_z(w) = zw$.

(a) Show that for each $z \in U$ the function T_z is in $S_{\mathbf{C}}$.

(b) Show that the function $\varphi(z) = T_z$ defines a homomorphism from U to $S_{\mathbf{C}}$.

(c) What are the orbits of U?

Exercise 2. Let G be the group of rotational symmetries of a cube.

(a) Let X be the set of lines connecting vertices of the cube. There is one line for every pair of distinct vertices. How many elements does X have?

(b) How many elements does G have?

(c) If a and b are vertices of the cube, we write $\overline{ab}$ for the line connecting a and b. Notice that $\overline{ab} = \overline{ba}$. For $g \in G$, define a function $T_g : X \to X$ by the rule

$$T_g(\overline{ab}) = \overline{g(a)g(b)}.$$

(d) (Skip this step in class) Show that $T_g \in S_X$ and the function $\varphi : G \to T_X$ defined by $\varphi(g) = T_g$ is a homomorphism.

(e) Draw a picture illustrating the orbits of this group action.

(f) Find the size of each orbit and the size of the stabilizer of an element of that orbit. Verify that these satisfy the expected relationship.

(g) Let $Y \subset X$ be the set of long diagonals of the cube. Show that for each $g \in G$, the function $T_g|_Y$ is in S_Y.

(h) (Skip this step in class) Show that the function $\psi : G \to S_Y$ defined by $\psi(g) = T_g|_Y$ is a homomorphism.

(i) Prove that ψ is an isomorphism.

Exercise 3. Let $H = S_4$ be the group of symmetries of the set $\{1, 2, 3, 4\}$ and let G be the **stabilizer** of 4 in S_4. This means that G is the subgroup of all elements g in S_4 such that $g(4) = 4$.

(a) Show that G is isomorphic to S_3.

(b) For each $g \in G$, let $L_g : H \to H$ be the function

$$L_g(h) = gh.$$

Why is $L_g \in S_H$?

(c) Let $\varphi : G \to S_H$ be the function $\varphi(g) = L_g$. Show that φ is a homomorphism.

(d) Draw a picture showing the orbits of the G action on H. What features do you observe about these orbits?

(e) For each $x \in H$, let O_x be the orbit of x under the action of G. For arbitrary $x, y \in H$, construct a bijection between O_x and O_y.