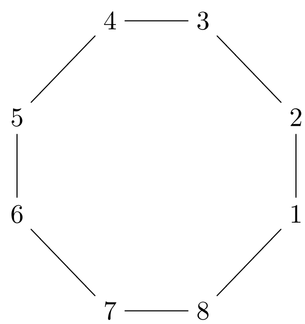
Math 3140 — Fall 2012
Handout #5

Exercise 1. In this problem we will study the regular octagon below, with vertices labelled as indicated:



Let X be the set of lines connecting vertices of a regular octagon. Note that only some of these lines are drawn above. We will need a convenient notation for the elements of X so we will write $\overline{ab}$ to designate the line connecting vertex a to vertex b.

1. How many elements does X have?

2. Let D_8 be the group of rigid symmetries of a regular octagon. How many elements does D_8 have? In order to talk about elements of D_8, we will use ρ to denote rotation by $\frac{\pi}{4}$ and let τ denote reflection through the line connecting vertex 1 to vertex 5.

   Using this notation, list all of the elements of D_8.

3. For each $\sigma \in D_8$, let $T_\sigma : X \to X$ be the following function:
$$T_\sigma(\overline{ab}) = \overline{\sigma(a)\sigma(b)}.$$
   Show that $T_\sigma \in S_X$.

4. In order to get a feel for what T_σ does, draw the lines $T_\rho(\overline{24})$, $T_\tau(\overline{17})$, and $T_{\rho^2\tau}(\overline{25})$. Do more examples until you feel you understand how T_σ works.

5. Show that the function $\varphi : D_8 \to S_X$ defined by $\varphi(\sigma) = T_\sigma$ is a homomorphism.

6. Draw a picture with one vertex for each element of X and an arrow from $\overline{ab}$ to $\overline{cd}$ whenever there is a $\sigma \in D_8$ such that $T_\sigma(\overline{ab}) = \overline{cd}$.

7. How many orbits does the action of D_8 on X have? What are the sizes of the orbits.

8. Let $X_1, \ldots, X_k$ denote the orbits of this action. Verify the formula
$$(\text{size of } X) = \sum_{i=1}^{k} (\text{size of } X_i).$$

9. Let x be an element of X. Let $G_x \subset D_8$ be the isotropy subgroup of x and let O_x be the orbit of x. How are the sizes of G_x and O_x related?