Math 3140 — Fall 2012

Handout #4

Exercise 1. Let X be the set $\{1,2,3\}^3$ of triples of integers between 1 and 3. Let $T_\sigma(i,j,k) = (\sigma(i), \sigma(j), \sigma(k))$ and let $U_\sigma(x_i, x_j, x_k) = (x_{\sigma(i)}, x_{\sigma(j)}, x_{\sigma(k)})$. Let $f : S_3 \to S_X$ and $g : S_3 \to S_X$ be the functions $f(\sigma) = T_\sigma$ and $g(\sigma) = U_\sigma$. Are these homomorphisms?

Solution. The function f is a homomorphism but g is not. For example, let's compare $g((12)(123)) = g(23) = U_{(23)}$ with $g(12)g(123) = U_{(12)}U_{(123)}$. We can compare these functions by looking at how they act on the element $(1,2,3) \in X$. We have

$$\begin{aligned} U_{(23)}(1,2,3) &= (1,3,2) \\ U_{(12)}U_{(123)}(1,2,3) &= U_{(12)}(2,3,1) = (3,2,1). \end{aligned}$$

These are not the same, so the functions $U_{(12)}U_{(123)}$ and $U_{(23)}$ are different.

On the other hand, we can verify that $f(\sigma\tau) = T_{\sigma\tau}$ is the same as $f(\sigma)f(\tau) = T_\sigma T_\tau$. Here is the verification: To see that two functions are the same we have to check that they have the same effect when applied to any element (i,j,k) of X. We have

$$\begin{aligned} T_\sigma T_\tau(i,j,k) &= T_\sigma(T_\tau(i,j,k)) = T_\sigma(\tau(i), \tau(j), \tau(k)) = (\sigma\tau(i), \sigma\tau(j), \sigma\tau(k)) \\ T_{\sigma\tau}(i,j,k) &= (\sigma\tau(i), \sigma\tau(j), \sigma\tau(k)). \end{aligned}$$

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