Math 3140 — Fall 2012

Handout #3

Exercise 1. Let $X = \{1, 2, 3\}^3$ be the set of all triples (i, j, k) where i,j, and k are integers between 1 and 3.

1. How many elements does X have?

2. Let S_X be the group of symmetries of X. How many elements does S_X have?

3. For each $\sigma \in S_3$, let $T_\sigma : X \to X$ be the function defined by the following rule

$$T_\sigma(x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}).$$

   Show that $T_\sigma \in S_X$. (Recall, S_X is defined to be the set of bijective functions from X to itself.)

4. Let $\varphi : S_3 \to S_X$ be the function

$$\varphi(\sigma) = T_\sigma.$$

   Show that φ is a homomorphism.

5. Is φ surjective?

6. Is φ injective?

7. φ is known as a **group action** because it permits us to use the elements of S_3 to **act on** X. Draw a picture of the elements of X with arrows indicating how the permutation $T_{(123)}$ acts on X.

8. Now draw a picture showing how $T_{(12)}$ acts on X.

9. Finally, draw a picture that shows all of the T_σ acting on X, for every $\sigma \in S_3$. What features can you recognize in this picture?

10. Observe that there are "components" in your picture above that are not connected by any arrows. These are known as **orbits** of the group action. How many orbits are there? How large is each orbit?

11. Let $X_1, \ldots, X_k$ be the orbits of the group action φ. Verify the following formula:

$$(\text{size of } X) = \sum_{i=1}^{k} (\text{size of } X_i).$$

    Come up with a reason why this formula should be true.

12. Consider the element $(1, 1, 2) \in X$. Let $G \subset S_3$ be the collection of $\sigma \in S_3$ such that $T_\sigma(1, 1, 2) = (1, 1, 2)$. Show that G is a subgroup of S_3. This is known as the **isotropy subgroup** or **stabilizer** of $(1, 1, 2)$.

13. For each element $x = (i, j, k)$ of X, let G_x be its isotropy subgroup of S_3. Show that G_x is a subgroup of S_3.

14. Choose an element of each orbit of the S_3 action on X and compute the size of its isotropy subgroup of S_3.

15. Let O_x denote the orbit of x under the action φ. Formulate a conjecture relating the sizes of O_x, G_x, and S_3.