Math 3140 — Fall 2012
Handout #2

Exercise 1. Let G be the set of pairs (a, b) where $a \in \mathbf{Z}/3\mathbf{Z}$ and $b \in \mathbf{Z}/4\mathbf{Z}$. Give G the following operation:

$$(a, b) + (a', b') = (a + a', b + b').$$

This is a group.

(i) What is the identity element of G?

(ii) What is the inverse of (a, b) in G?

(iii) Verify that the operation defined above is associative.

(iv) Compute the number of elements in $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.

(v) Compute the number of elements of $\mathbf{Z}/12\mathbf{Z}$.

(vi) Draw the orbit of $(1, 1)$ in G.

(vii) Define a function $\varphi : \mathbf{Z}/12\mathbf{Z} \to G$ by the rule

$$\varphi(x) = (x \bmod 3, x \bmod 4).$$

Show that φ is well defined.

(viii) Show that φ is a homomorphism.

(ix) Show that φ is injective.

(x) Conclude that φ is an isomorphism.

Exercise 2. Let G be the set of pairs (a, b) where $a \in \mathbf{Z}/4\mathbf{Z}$ and $b \in \mathbf{Z}/6\mathbf{Z}$. Give G the following operation:

$$(a, b) + (a', b') = (a + a', b + b').$$

This is a group.

(i) What is the identity element of G?

(ii) What is the inverse of (a, b) in G?

(iii) Verify that the operation defined above is associative.

(iv) Draw the orbit of $(1, 1)$ in G.

(v) Define a function $\varphi : \mathbf{Z}/24\mathbf{Z} \to G$ by the rule

$$\varphi(x) = (x \bmod 4, x \bmod 6).$$

Show that φ is well defined.

(vi) Show that φ is a homomorphism.

(vii) Is φ injective? Justify your answer.

(viii) Is φ surjective? Justify your answer.

(ix) Is G isomorphic to $\mathbf{Z}/24\mathbf{Z}$?

(x) What is the kernel of φ?

(xi) Find two non-trivial groups A and B such that $\mathbf{Z}/24\mathbf{Z}$ is isomorphic to $A \times B$.

Definition 1. Suppose that G and H are groups, the product of G and H is the set of pairs (g, h) where $g \in G$ and $h \in H$ with the group law

$$(g, h)(g', h') = (gg', hh').$$

Exercise 3. If p is a prime number, is $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$ isomorphic to $\mathbf{Z}/p^2\mathbf{Z}$?

Exercise 4. Formulate a conjecture about when $\mathbf{Z}/mn\mathbf{Z}$ is isomorphic to $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$.

Exercise 5. (i) Is D_n isomorphic to the product of $\mathbf{Z}/n\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$?

(ii) Is S_n isomorphic to $A_n \times \{\pm 1\}$?

Exercise 6. Let G be the group of rigid symmetries of the following pattern:

$$\cdots \text{EEEEEEEEEEEEEEEEEEEEEEEEE} \cdots$$

The dots mean that the pattern continues to infinity in both directions. Describe G as the product of two familiar groups.