Math 3140 — Fall 2012

Handout #2

Exercise 1. Let G be the set of pairs (a, b) where $a \in \mathbf{Z}/3\mathbf{Z}$ and $b \in \mathbf{Z}/4\mathbf{Z}$. Give G the following operation:

$$(a \bmod 3, b \bmod 4) + (a' \bmod 3, b' \bmod 4) = ((a + a') \bmod 3, (b + b') \bmod 4).$$

This is a group.

(i) What is the identity element of G?

Solution. The identity is $(0, 0)$ becuase $(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b)$. The addition law is commutative so $(0, 0)$ is also a right identity (you can also verify this directly). ☐

(ii) What is the inverse of (a, b) in G?

Solution. The inverse of (a, b) is $(-a \bmod 3, -b \bmod 3)$ because

$$(a, b) + (-a \bmod 3, -b \bmod 3) = ((a - a) \bmod 3, (b - b) \bmod 4) = (0, 0).$$

☐

(iii) Verify that the operation defined above is associative.

Solution. We have

$$\begin{aligned}
((x, y) + (x', y')) + (x'', y'') &= (x + x', y + y') + (x'', y'') && \text{definition of addition in } \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \\
&= ((x + x') + x'', (y + y') + y'') && \text{definition of addition in } \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \\
&= (x + (x' + x''), y + (y' + y'')) && \text{associativity in } \mathbf{Z}/3\mathbf{Z} \text{ and in } \mathbf{Z}/4\mathbf{Z} \\
&= (x, y) + (x' + x'', y' + y'') && \text{definition of addition in } \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \\
&= (x, y) + ((x', y') + (x'', y'')) && \text{definition of addition in } \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}.
\end{aligned}$$

☐

(iv) Compute the number of elements in $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.

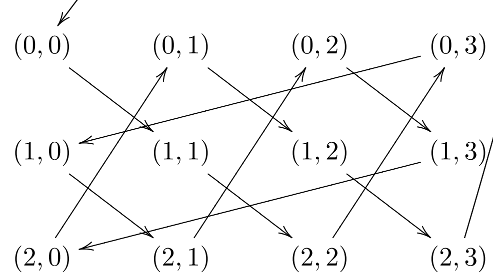
Solution. For each element of $\mathbf{Z}/3\mathbf{Z}$ we have one element of $\mathbf{Z}/4\mathbf{Z}$ so the total number of elements is $3 \cdot 4 = 12$. ☐

(v) Compute the number of elements of $\mathbf{Z}/12\mathbf{Z}$.

Solution. There are 12. ☐

(vi) Draw the orbit of $(1, 1)$ in G.

Solution.



☐

(vii) Define a function $\varphi : \mathbf{Z}/12\mathbf{Z} \to G$ by the rule

$$\varphi(x) = (x \bmod 3, x \bmod 4).$$

Show that φ is well defined. Show, in other words, that if $x \bmod 12 = y \bmod 12$ that $\varphi(x) = \varphi(y)$.

Solution. If $x \bmod 12 = y \bmod 12$ then $x - y$ is a multiple of 12: that is, $x - y = 12k$ for some integer k. Then

$$\begin{aligned}\varphi(x) = \varphi(y + 12k) &= ((y + 12k) \bmod 3, (x + 12k) \bmod 4) \\ &= (y \bmod 3, y \bmod 4) = \varphi(y)\end{aligned}$$

because $(y+12k) \bmod 3 = y \bmod 3$ (since $(y+12k)-y$ is divisible by 3) and $(y+12k) \bmod 4 = y \bmod 4$ (since $(y + 12k) - y$ is divisible by 4). $\square$

(viii) Show that φ is a homomorphism.

Solution. We have to check that $\varphi(x + y) = \varphi(x) + \varphi(y)$. We have

$$\begin{aligned}\varphi(x + y) &= ((x + y) \bmod 3, (x + y) \bmod 4) && \text{definition of } \varphi \\ &= (x \bmod 3 + y \bmod 3, x \bmod 4 + y \bmod 4) && \text{definition of modular addition} \\ &= (x \bmod 3, x \bmod 4) + (y \bmod 3, y \bmod 4) && \text{definition of addition in } \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \\ &= \varphi(x) + \varphi(y) && \text{definition of } \varphi\end{aligned}$$

$\square$

(ix) Show that φ is injective.

Solution. We have to show that we can only have $\varphi(x) = \varphi(y)$ if $x = y$ in $\mathbf{Z}/12\mathbf{Z}$. Suppose that $\varphi(x) = \varphi(y)$ for some x and y in $\mathbf{Z}/12\mathbf{Z}$. This means that

$$(x \bmod 3, x \bmod 4) = (y \bmod 3, y \bmod 4)$$

so $x \bmod 3 = y \bmod 3$ and $x \bmod 4 = y \bmod 4$. By definition of equality modulo 3, this means that $x - y$ is divisible by 3, and by definition of equality modulo 4, this means that $x - y$ is divisible by 4. Therefore $x - y$ is divisible by $\operatorname{lcm} 3, 4 = 12$. But now by definition of equality modulo 12, this means that $x \bmod 12 = y \bmod 12$. That is, x and y are the same element of $\mathbf{Z}/12\mathbf{Z}$. $\square$

(x) Conclude that φ is an isomorphism.

Solution. An isomorphism is a bijective homomorphism, and we already know that φ is an injective homomorphism from our work above. But we also know that $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ and $\mathbf{Z}/12\mathbf{Z}$ have the same size and an injection between sets of the same size is also a surjection. Since φ is an injection it is therefore also a surjection, hence a bijection. Thus φ is an isomorphism. $\square$

Exercise 2. Let G be the set of pairs (a, b) where $a \in \mathbf{Z}/4\mathbf{Z}$ and $b \in \mathbf{Z}/6\mathbf{Z}$. Give G the following operation:

$$(a, b) + (a', b') = (a + a', b + b').$$

This is a group.

(i) What is the identity element of G?

(ii) What is the inverse of (a, b) in G?

(iii) Verify that the operation defined above is associative.

(iv) Draw the orbit of $(1,1)$ in G.

(v) Define a function $\varphi : \mathbf{Z}/24\mathbf{Z} \to G$ by the rule

$$\varphi(x) = (x \bmod 4, x \bmod 6).$$

Show that φ is well defined.

(vi) Show that φ is a homomorphism.

(vii) Is φ injective? Justify your answer.

(viii) Is φ surjective? Justify your answer.

(ix) Is G isomorphic to $\mathbf{Z}/24\mathbf{Z}$?

(x) What is the kernel of φ?

(xi) Find two non-trivial groups A and B such that $\mathbf{Z}/24\mathbf{Z}$ is isomorphic to $A \times B$.

Definition 1. Suppose that G and H are groups, the product of G and H is the set of pairs (g,h) where $g \in G$ and $h \in H$ with the group law

$$(g,h)(g',h') = (gg',hh').$$

Exercise 3. If p is a prime number, is $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$ isomorphic to $\mathbf{Z}/p^2\mathbf{Z}$?

Exercise 4. Formulate a conjecture about when $\mathbf{Z}/mn\mathbf{Z}$ is isomorphic to $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$.

Exercise 5. (i) Is D_n isomorphic to the product of $\mathbf{Z}/n\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$?

(ii) Is S_n isomorphic to $A_n \times \{\pm 1\}$?

Exercise 6. Let G be the group of rigid symmetries of the following pattern:

$$\cdots \mathrm{EEEEEEEEEEEEEEEEEEEEEEEEE} \cdots$$

The dots mean that the pattern continues to infinity in both directions. Describe G as the product of two familiar groups.