Math 3140 — Fall 2012
Exam #3

Due Wednesday, December 19. Work alone. You may consult any textual references you want (including textbooks and the internet), but you may not discuss the exam with anyone (neither in person, nor over the phone or internet) except me. You are not allowed to use a calculator or computer to help with calculations.

Problem 1. Compute $3^{10202} \pmod{101}$. (Hint: 101 is prime.)

Problem 2. Let $\varphi : \mathbf{C}^* \to \mathbf{C}^*$ be the function $\varphi(z) = z^3$.

(a) Show that φ is a homomorphism.

(b) Is φ surjective?

(c) What is the kernel of φ?

(d) Is φ an isomorphism?

Problem 3. Let Q be the group whose elements are $\{\pm 1, \pm i, \pm j, \pm k\}$ with the following multiplication table:

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

(a) What is the identity of Q?

(b) List the inverses of each of the elements of Q.

(c) Is Q abelian?

(d) Find the orders of all the elements of Q.

(e) Show that Q is not isomorphic to any of the following groups:

  (i) $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$

  (ii) D_4

(f) Find all subgroups of Q.

(g) Which of the subgroups in your list are normal?

(h) Match each normal subgroup N of Q with the group listed below that is isomorphic to Q/N.

  (i) Q

  (ii) $\{1\}$

  (iii) $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$

  (iv) $\mathbf{Z}/4\mathbf{Z}$

  (v) $\mathbf{Z}/2\mathbf{Z}$

Justify your answers.

Problem 4. Let G be the set of invertible 2×2 matrices with entries in $\mathbf{F}_2$.

(a) Show that G is a group in which the operation is matrix multiplication.

(b) How many elements does G have?

(c) Let X be the set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \in \mathbf{F}_2$. How many elements does X have?

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of G and $\begin{pmatrix} x \\ y \end{pmatrix}$ is an element of X, define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

(This is the standard formula for matrix multiplication.) This defines an action of G on X.

(d) What are the orbits of this action?

(e) What is the stabilizer subgroup of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

(f) Find an isomorphism between G and S_3.

Problem 5. Suppose that G is a finite group with n elements. Let p be a prime number. Assume that n is not divisible by p. Show that the only group homomorphism $\varphi : G \to \mathbf{Z}/p\mathbf{Z}$ is given by $\varphi(g) = 0$ for all $g \in G$.

Problem 6. Construct a field with 169 elements. Prove that your construction actually gives a field and that it has the right number of elements.

Problem 7. Let[1] $\mathbf{H}$ be the ring of all symbols $a + bi + cj + dk$ where $a, b, c, d \in \mathbf{R}$ with the following addition and ←[1]
multiplication laws:

$$\begin{aligned} (a + bi + cj + dk) + (a' + b'i + c'j + d'k) &= (a + a') + (b + b')i + (c + c')j + (d + d')k \\ (a + bi + cj + dk)(a' + b'i + c'j + d'k) &= (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i \\ &\quad + (ac' - bd' + ca' + db')j + (ad' + bc' - cb' + da')k \end{aligned}$$

(a) Is $\mathbf{H}$ commutative? Justify your answer.

(b) Each ring in the first column is isomorphic to a ring in the second column. Determine which, and say what the isomorphisms are. You should give reasoning for why your answers are the correct ones, but you do not have to verify in complete detail that your constructions are isomorphisms.

(i) $\mathbf{H}$

(ii) $\mathbf{C} \times \mathbf{C}$

(iii) $\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$

(I) the set of matrices $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$ with $a, b, c, d \in \mathbf{R}$

(II) the set of matrices $\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$ with $a, b, c, d \in \mathbf{R}$

(III) the set of matrices $\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}$ with $a, b, c, d \in \mathbf{R}$

Problem 8. Prove the ring $\mathbf{R}[x]/(x^3 - x)$ is isomorphic as a ring to $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

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[1] This problem has been changed completely!