Finding kernels and images — find a good basis.

Notes: • changing basis on dom(T) does not change im(T) — so if you only need im(T) then do column operations to find a basis

[$\mathrm{im}[T]^{\beta}_{\alpha} = \mathrm{im}[T]^{\beta}_{\alpha'}$]

• changing basis on codom(T) does not change ker(T) — so if you only need ker(T) then do row operations to find ker(T)

[$\ker[T]^{\beta}_{\alpha} = \ker[T]^{\beta'}_{\alpha}$]

• don't have to go all the way to $\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$ to find image or kernel — just go far enough that you can see what image and kernel are.

Example
$$\begin{pmatrix} 2 & 1 & -1 & 2 \\ 3 & 1 & 2 & 1 \\ 1 & -2 & 17 & -9 \end{pmatrix}$$

Find image:
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & -1 \\ 5 & -2 & 15 & -5 \end{pmatrix}$$

so $(1,1,-2)$ and $(0,1,5)$ are a basis for image.

Find kernel: (already know $\dim(\ker) = 2$)
$$\begin{pmatrix} 2 & 1 & -1 & 2 \\ 1 & 0 & 3 & -1 \\ 5 & 0 & 15 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -1 & 2 \\ 1 & 0 & 3 & -1 \\ 1 & 0 & 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 5 & 0 \\ 1 & 0 & 3 & -1 \end{pmatrix}$$

so if $(x,y,z,w) \in \ker(T)$ then x, z can be chosen arbitrarily

and
$$y = -4x - 5z$$
$$w = x - 3z$$

so kernel is $\operatorname{span}((1,-4,0,1),(0,-5,1,3))$

Find a system of equations defining image

[Need 1 equation since im has ~~dim~~ = 2.]

$$\operatorname{im} = \operatorname{im}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -2 & 5 \end{pmatrix}$$

$$= \operatorname{im}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -7 & 5 \end{pmatrix}$$

$$z = -7x + 5y$$

$$\operatorname{im}(T) = \ker\begin{pmatrix} -7 & 5 & -1 \end{pmatrix}$$

Is $(1,2,3)$ in the image? [Does system of eqns. have solution]

$$\left[\text{is } (1,2,3) \text{ in span}\left(\begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-2\end{pmatrix}, \begin{pmatrix}-1\\2\\17\end{pmatrix}, \begin{pmatrix}2\\1\\-9\end{pmatrix} \right) = \text{span}\left(\begin{pmatrix}1\\1\\-2\end{pmatrix}, \begin{pmatrix}0\\1\\5\end{pmatrix} \right) \right]$$

$$\text{is rank}\left(M \;\middle|\; \begin{matrix}1\\2\\3\end{matrix} \right) > \text{rank}(M)\ ?$$

yes: $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \text{im}(T)$

no: $\begin{pmatrix}1\\2\\3\end{pmatrix} \notin \text{im}(T)$

$$\text{same as rank}\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 2\\ -2 & 5 & 3\end{pmatrix}$$

$$= \text{rank}\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ -2 & 5 & 5\end{pmatrix} = 2$$

so $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \text{im}(M)$.

Find a vector $v = (x, y, z, w)$ such that $T(v) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$?

[find a solution]

Solve by doing row ops simultaneously to

$[T]$ and v.

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 1 & -2 & 17 & -9 & 3 \end{array}\right)$$

$Mv = w$

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 1 \\ 1 & 0 & 3 & -1 & 1 \\ 5 & 0 & 15 & -5 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 1 \\ 1 & 0 & 3 & -1 & 1 \\ 1 & 0 & 3 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 4 & 1 & 5 & 0 & 3 \\ 1 & 0 & 3 & -1 & 1 \end{array}\right)$$

$AMv = Aw$

so $v = (0, 3, 0, -1)$ works.

Find $\underline{\underline{\text{all}}}$ vectors w such that $T(w) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. [find all solutions]

Note: $T(w) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \iff T(w - v) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff w - v \in \ker(T)$

so $T^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = v + \ker(T) = v + \operatorname{span}\left(\vdots , \vdots \right)$

$= \left\{ v + u \;\middle|\; u \in \ker(T) \right\}$

$= \left\{ \begin{pmatrix} x \\ 3+y \\ z \\ -1+w \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(T) \right\}$

$= \left\{ \begin{pmatrix} x \\ 3-4x-5z \\ -1+x+3z \end{pmatrix} \;\middle|\; x, y \in \mathbb{F} \right\}$