let $V = M_{3\times 3}(\mathbb{R})$ = 2×2 matrices over $\mathbb{R}$

let $W = \left\{ M \in V \;\middle|\; \text{sums of all rows and columns are the same} \right\}$.

$$= \left\{ \begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & \to c \\ a_{21} & a_{22} & a_{23} & \to c \\ a_{31} & a_{32} & a_{33} & \to c \\ \hline \downarrow & \downarrow & \downarrow & \\ c & c & c & \end{array} \;:\; \begin{array}{l} a_{11} + a_{12} + a_{13} \\ = a_{21} + a_{22} + a_{23} \\ = a_{31} + a_{32} + a_{33} \end{array} \begin{array}{l} = a_{11} + a_{12} + a_{13} \\ = a \\ \end{array} \right.$$

— how can we show this is a subspace?

- check the axioms?
- notice it is the kernel of a linear transformation:

$$U : V \longrightarrow \mathbb{R}^{6^2} = \mathbb{R}^{36}$$

$$U(M) = (\text{all differences of sums along rows and columns})$$

this is pretty redundant — could use a map $V \to \mathbb{R}^5$

$$T(M) = (\text{row1} - \text{row2}, \text{row1} - \text{row3}, \text{row1} - \text{col1}, \text{row1} - \text{col2}, \text{row1} - \text{col3})$$

– compute $\dim W$

notice that $\dim(\operatorname{im}(T)) \le 5$ since $\operatorname{im}(T) \subseteq \mathbb{R}^5$

but $\dim \ker(T) + \dim \operatorname{im}(T) = \dim V = 9$

so $\dim \ker(T) \ge 9 - 5 = 4$!

– find a basis for W?

- start listing matrices until you have a basis

$$\underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}_{N_1}, \underbrace{\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}}_{N_2}, \underbrace{\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}}_{N_3}, \underbrace{\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}}_{N_4}$$

just put one non-zero entry in each row and each column!

need to check this is actually linearly independent and spanning.

spanning seems hard, but can prove linear independence using these entries:

$$\begin{pmatrix} \circ & \circ & \\ & & \\ \circ & \circ & \end{pmatrix}$$

if $\sum a_i N_i = 0$ then

$$\sum a_i N_i = \begin{pmatrix} a_1 & a_2 & ? \\ ? & ? & ? \\ a_4 & a_3 & ? \end{pmatrix}$$ so $a_1 = a_2 = a_3 = a_4 = 0$.

So $\dim W \geq 4$ since it contains 4 linearly independent vectors.

Therefore $\dim W = 4$ and w_1, w_2, w_3, w_4 is a basis.

We used a useful fact:

<u>Lemma</u> suppose $\varphi : V \longrightarrow W$ is a linear transformation, $v_1, \ldots, v_n \in V$ are lin. dep. Then $\varphi(v_1), \ldots, \varphi(v_n)$ is linearly dependent too.

<u>Proof</u> S dependent means

$$\sum_{i=1}^{n} a_i v_i = \vec{0}$$ for some distinct vectors $v_1, \ldots, v_n \in S$ and some scalars $a_1, \ldots, a_n \in F$.

hence $\vec{0} = \varphi(\vec{0}) = \varphi\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i \varphi(v_i)$ so

$\varphi(v_1), \ldots, \varphi(v_n)$ are dependent too.

<u>Corollary</u> If $v_1, \ldots, v_n \in V$ and $\varphi : V \longrightarrow W$ such that $\varphi(v_1), \ldots, \varphi(v_n)$ are independent then $v_1, \ldots, v_n$ are independent too.

In the example, we used

$$\varphi\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{11}, a_{12}, a_{32}, a_{31}).$$

$\varphi(N_i) = \vec{e}_i$ are lin. indep., so

N_i are lin. indep.

<u>Another example of this</u>

[this has been discussed before, but not using linear transformations]

$P_n(F)$ = polynomials of deg $\le n$ over F.

pick $s_0, \dots, s_n \in F$, pairwise distinct.

then let $$f_j(x) = \prod_{\substack{i=0,\dots,n \\ i \ne j}} \frac{(x - s_i)}{(s_j - s_i)}$$

Define $\varphi : P_n(F) \longrightarrow F^{n+1}$

$$\varphi(g(x)) = (g(s_0), \dots, g(s_n))$$

then $\varphi(f_j(x)) = \vec{e}_j$.

<u>Therefore</u>

1) $f_0, \dots, f_n$ are lin. indep.

2) $\operatorname{im} \varphi \supseteq \operatorname{span}\{\vec{e}_0, \dots, \vec{e}_n\} = F^{n+1}$

3) $\dim \operatorname{im} \varphi = n+1 \Rightarrow \dim \ker \varphi = 0$

4) hence $\ker \varphi = \{0\}$ (hence φ a bijection)