[These definitions have already been discussed.]

<u>Def</u> Suppose $T: V \longrightarrow W$ is a linear transformation.

$\ker(T) = N(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$ <u>kernel</u>

$\mathrm{im}(T) = R(T) = T(V) = \{T(\vec{v}) : \vec{v} \in V\}$ <u>image</u>

<u>Theorem</u> If V is a finite dimensional vector space and $T: V \longrightarrow W$ is a linear transformation then

$$\dim V = \dim \ker(T) + \dim \mathrm{im}(T).$$

<u>Example</u> let $V = P_n(\mathbb{R})$ and let $T: V \longrightarrow V$ be the linear transformation $T(f) = f'$.

<u>Claim</u> if $n \geq 0$, $\mathrm{im}(T) = P_{n-1}(\mathbb{R})$

$\ker(T) = P_0(\mathbb{R})$.

[convention: $P_{-1}(\mathbb{R}) = \{0\}$.]

<u>Check</u>: <u>image</u>: $\deg(f') < \deg(f)$ so $f' \in P_{n-1}(\mathbb{R})$

if $g \in P_{n-1}(\mathbb{R})$ then let $f(x) = \int_0^x g(x)\,dx$.

Then $f(x) \in P_n(\mathbb{R})$ and $f'(x) = g(x)$

by fund. thm. of calculus

hence $\mathrm{im}(T) = P_{n-1}(\mathbb{R})$.

<u>kernel</u>: $f'(x) = 0 \iff f$ constant $\iff f \in P_0(\mathbb{R})$.

$\dim P_{n-1}(\mathbb{R}) = n$

$\dim P_0(\mathbb{R}) = 1$

$\dim P_n(\mathbb{R}) = n+1$

so $\dim \ker(T) + \dim \operatorname{im}(T) = \dim P_n(\mathbb{R})$, as the theorem says.

<u>Thm</u> If V is a vector space and $S \subseteq V$ is linearly independent then there is some $S' \supseteq V$ such that S' is a basis of V.

[this was proved before — just recall the statement; I usually recall the idea of the proof out loud.

<u>Lemma</u> If $T: V \longrightarrow W$ is a linear transformation and $S \subseteq V$ then $T(\operatorname{span}(S)) = \operatorname{span}(T(S))$.

<u>Proof</u>

$$\begin{aligned} w \in \operatorname{span}(T(S)) &\iff w = \sum_{i=1}^{n} a_i T(v_i) \quad \text{for some } v_1, \ldots, v_n \in S \\ &\iff w = T\left(\sum_{i=1}^{n} a_i v_i\right) \quad \text{for some } v_1, \ldots, v_n \in S \\ &\iff w \in T(\operatorname{span}(S)). \end{aligned}$$

Proof of rank-nullity theorem

Suppose $T: V \longrightarrow W$ is a linear transformation.

Let $v_1, \dots, v_n \in \ker(T)$ be a basis.

then $\{v_1, \dots, v_n\}$ are linearly independent in V,

so can be extended to a basis $\{v_1, \dots, v_n, w_1, \dots, w_m\}$ of V. By defn. of dimension, we have:

$$\dim \ker(T) = n$$

$$\dim V = n+m$$

We want to prove that $\dim \operatorname{im}(T) = m$. We will show that $T(w_1), \dots, T(w_m)$ are a basis of $\operatorname{im}(T)$. This has two parts:

spanning: $\operatorname{span}\{T(w_1), \dots, T(w_m)\}$

$$= \operatorname{span}\{\underbrace{T(v_1), \dots, T(v_n)}_{0 \text{ since } v_1, \dots, v_n \in \ker(T)}, T(w_1), \dots, T(w_m)\}$$

$$= T(\operatorname{span}\{v_1, \dots, v_n, w_1, \dots, w_m\}) \quad \text{[by Lemma]}$$

$$= T(V) \quad \text{[since } \{v_1, \dots, v_{n+m}\} \text{ is a basis of } V\text{]}$$

$$= \operatorname{im}(T) \quad \text{[defn. of } \operatorname{im}(T)\text{]}$$

linearly independent:

suppose $\sum_{i=1}^{m} a_i T(w_i) = 0$

then $T\left(\sum_{i=1}^{m} a_i w_i\right) = 0$

so $\sum_{i=1}^{m} a_i w_i \in \ker(T)$ [by defn. of ker(T)]

so $\sum_{i=1}^{m} a_i w_i = \sum_{j=1}^{n} b_j v_j$ [since $\{v_1, \dots, v_n\}$ are a basis of ker(T)]

so $\sum_{j=1}^{n} b_j v_j - \sum_{j=1}^{m} a_i w_i = \vec{0}$

But $\{v_1, \dots, v_n, w_1, \dots, w_m\}$ are linearly indep.

so $a_i = b_j = 0$ for all i, j.

Hence $T(w_1), \dots, T(w_m)$ are independent.