Agenda

0) sign in / attendance

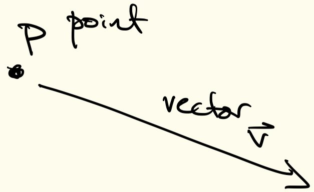
1) exercises –

2) questions

3) vector spaces
   - scalar field
   - examples of vector spaces
   - proving something is a vector space.

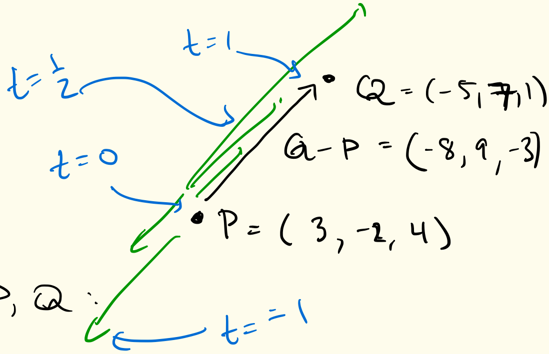
What is the difference between vector to the point and the point itself?

One refers to position, the other to motion. these are different species of objects.



Exercises

Parameterizations:

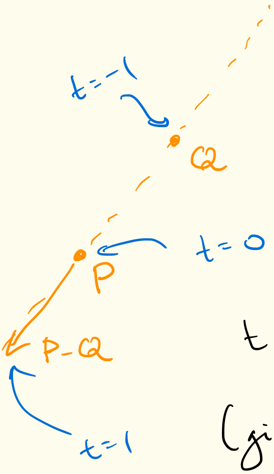


- a line through P, Q:

$$\{P + t(Q-P) : t \in \mathbb{R}\} = \{(3,-2,4) + t(-8,9,-3) : t \in \mathbb{R}\}$$

$$\{Q + u(P-Q) : t \in \mathbb{R}\}$$

$$\{P + t(P-Q) : t \in \mathbb{R}\}.$$



t is a coordinate on the line

(gives you a way of describing points — will be important when we talk about basis)

- a plane through P, Q, R

$$\{P + s(Q-P) + t(R-P) : s,t \in \mathbb{R}\}$$

or other variants.

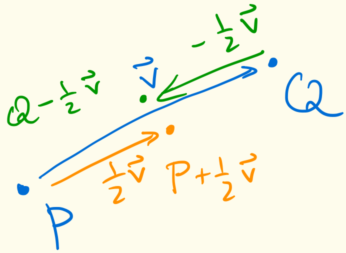
$$\{P + s(Q-P) + t(R-Q) : s,t \in \mathbb{R}\}$$

6) If P, Q are points, describe point that is halfway between.

$\vec{v} = Q - P$ = vec. from P to Q.

travel half as far:

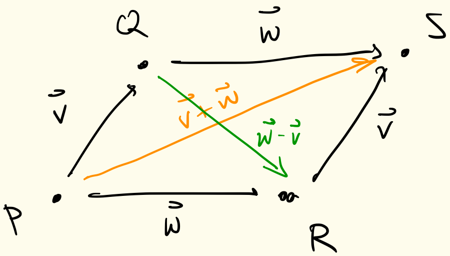
$P + \frac{1}{2}\vec{v}$



or $\vec{w} = P - Q = -\frac{1}{2}\vec{v}$

$Q + \frac{1}{2}\vec{w}$.

7) In a parallelogram, diagonals bisect each other (meet at their midpoint)



need to check midpoints are the same.

midpoints of diagonals are

$$P + \frac{1}{2}(\vec{v} + \vec{w}) \qquad\qquad Q + \frac{1}{2}(\vec{w} - \vec{v})$$

but $\quad \vec{v} = Q - P$

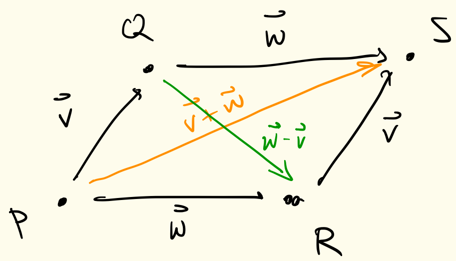
$Q = P + \vec{v} \qquad$ so

$$Q + \frac{1}{2}(\vec{w} - \vec{v}) = P + \vec{v} + \frac{1}{2}\vec{w} - \frac{1}{2}\vec{v}$$

$$= P + \vec{v} - \frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = P + \left(1 - \frac{1}{2}\right)\vec{v} + \frac{1}{2}\vec{w}$$

$$= P + \frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}$$

$$= P + \frac{1}{2}(\vec{v} + \vec{w}) \quad \text{is the same point.}$$



need to check midpoints are the same.

Here we are studying the vector space V of linear motions in the plane. We used some basic facts about these vectors in our discussion:

[VS3] 0) There is a special element of V called $\vec{0}$.

1) If $\vec{v}$, $\vec{w} \in V$, there is a vector $\vec{v}+\vec{w} \in V$ (that represents the linear motion equivalent to $\vec{v}$ then $\vec{w}$)

2) If $\vec{v} \in V$ and $\lambda \in \mathbb{R}$ then there is a vector $\lambda\vec{v}$ (representing linear motion by λ times as far in direction of $\vec{v}$).

must supply these before you can start a proof that V is a vector sp.

[VS2] 3) $\forall\ \vec{u}, \vec{v}, \vec{w} \in V$, $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

[VS1] 4) $\forall\ \vec{u}, \vec{v} \in V$, $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

[VS3] 5) $\forall\ \vec{v} \in V$, $\vec{0}+\vec{v} = \vec{v}$.

[VS4] 6) $\forall\ \vec{v} \in V$, $\exists\ \vec{w} \in V$, $\vec{v}+\vec{w} = \vec{0}$. $\left[\vec{w} = -\vec{v}\right]$

[VS5] 7) $\forall \vec{v} \in V$, $1\vec{v} = \vec{v}$.

[VS6] 8) $\forall \vec{v} \in V$, $\forall \lambda, \mu \in \mathbb{R}$, $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$.

[VS8] 9) $\forall \vec{v} \in V$, $\forall \lambda, \mu \in \mathbb{R}$, $(\lambda+\mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$.

[VS7] 10) $\forall \vec{u}, \vec{v} \in V$, $\forall \lambda \in \mathbb{R}$, $\lambda(\vec{u}+\vec{v}) = \lambda\vec{u} + \lambda\vec{v}$.

In general, a <u>real</u> <u>vector space</u> is a set V with the above structure.

What about multiplying vectors?

Examples of Vector Spaces

1) V = all ways of changing the ingredient list of a recipe.

2) $V = \mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$

$\vec{0} = (0, \dots, 0)$

(for $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$)

$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

$\lambda(x_1, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$ (for $\lambda \in \mathbb{R}$, $(x_1, \dots, x_n) \in \mathbb{R}^n$)

VS5 fails! ~~[or $= (2\lambda x_1, 2\lambda x_2, \dots, 2\lambda x_n)$]~~

Can $\mathbb{R}^n$ have other vector space structures?

(usually write $\mathbb{R}^n$ as columns rather than rows.)

We must prove that $\mathbb{R}^n$, with these operations is a vector space. This is skipped in the book so it becomes your responsibility!

[VS1] $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Proof Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$.

Therefore $\vec{u} = (x_1, \dots, x_n)$ for some $x_1, \dots, x_n \in \mathbb{R}$

and $\vec{v} = (y_1, \dots, y_n)$ for some $y_1, \dots, y_n \in \mathbb{R}$.

$$\vec{u} + \vec{v} = (x_1, \dots, x_n) + (y_1, \dots, y_n) \quad \text{(def of } \vec{u}, \vec{v} \text{ above)}$$

$$= (x_1 + y_1, \dots, x_n + y_n) \quad \text{(def. of + } \mathbb{R}^n\text{)}.$$

$$\left.\begin{aligned} \vec{v} + \vec{u} &= (y_1, \dots, y_n) + (x_1, \dots, x_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \end{aligned}\right\} \text{same reasons}$$

These are the same because $x_i + y_i = y_i + x_i$ since $+$ in $\mathbb{R}$ is commutative!

QED.

[VS7] $\forall \lambda \in \mathbb{R}, \forall \vec{u}, \vec{v} \in \mathbb{R}^n, \lambda(\vec{u}+\vec{v}) = \lambda\vec{u} + \lambda\vec{v}$.

Proof

What is the field F?

A field is a set with 2 special elements

$0 \neq 1$ $\quad 0, 1 \in F$ and two special operations $+, \cdot$

With a bunch of rules:

[basically $+, -, \times, \div$ behave as expected]

All linear algebra can be done using an arbitrary field in place of $\mathbb{R}$.

<u>Examples of fields</u>

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$\mathbb{Q} = \left\{\frac{m}{n} : m, n \text{ are integers}, n \neq 0\right\}$$

$$\mathbb{R}$$

$$\cancel{\mathbb{Z}}$$

$\mathbb{Z}/n\mathbb{Z}$ is a field iff n prime int.

<u>Example</u> let $V = \mathbb{R}$ with $\vec{0} = 0$

and $F = \mathbb{Q}$

$$\vec{u} + \vec{v} = u + v \qquad u, v \in \mathbb{R}$$

$$\lambda \vec{v} = \lambda v \qquad \text{if } \lambda \in \mathbb{Q},\ v \in \mathbb{R}$$

Then $\mathbb{R}$ is a $\mathbb{Q}$-vector space with these operations.