Math 3135 — Honors Linear Algebra

LIBR M498

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Why is linear algebra important?

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Math 204

There are 2 kinds of math problems

- those that can be turned into linear algebra problems and
- those that are impossible.

Math gets very hard very quickly when you go beyond linear equations so linear algebra is basically the only thing we can do.

$\Longrightarrow$ linear algebra is used everywhere.

- study symmetries by symmetries of vector spaces (linear algebra)
- google page rank (eigenvector)

- study solutions to polynomial eqns by studying vector spaces gen. by solutions

- diff. calculus = approximating functions by linear functions

- special relativity = change of variables in a vector space w/ a bilinear form of signature (1,3).

What is linear algebra?

$$ax + b = c$$
$$x = \frac{c-b}{a}$$

- The study of linear equations and their solutions.

$$a_1 x_1 + \cdots + a_n x_n = 0$$

where a_i are scalars and x_i are variables

- linear equations and the solutions to a system of linear equations can be <u>scaled</u> and <u>summed</u> so linear algebra becomes the study of <u>vector spaces</u>

- in this class we will study vector spaces from the beginning, how to build vector spaces, how to use them to analyze linear equations

Structure of this class

- Honors class $\Rightarrow$ high expectations, high workload.

  payoff = good foundation for more math classes, practice doing proofs, natural sequel to Math 2001

- plan to spend 9 hours per week outside of class. (should actually do this for all of your classes)

- plan to come to office hours regularly (Math 204 — see website for calendar)

- plan to do self-directed work

- do the daily homework assignments.

- Grading is standards-based:

all grades on a 0–5 scale

$0 = F$
$1 = D$
$2 = C$
$3 = B$
$4 = A$
$5 = A!$

class has 8–10 areas and you will get a grade in each area. Final grade is (roughly) the lowest of these

A = average M = min

(actually $G = M + \max\left\{0, \frac{A-M}{4-M}\right\}$)

in each topic, grade is the highest 2 (from different areas) of

1) quiz and exam scores
(each topic quizzed in class, once out of class twice)
2) homework/take-home exam scores
(can be revised)
3) portfolio scores.
(either D2L or in person)

a portfolio on a topic should include

- summary of all relevant definitions
- statements of relevant theorems
- worked problems demonstrating the relevant skills (of own selection)

should be 3-5 pages long, probably.

— Academic integrity

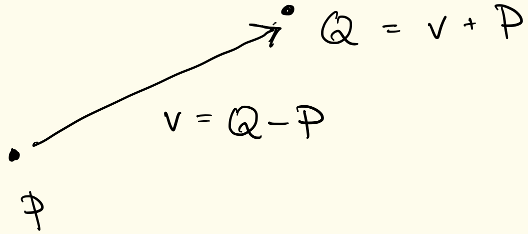
Basic rule: everything you write and submit as your own must be a reflection of your own understanding

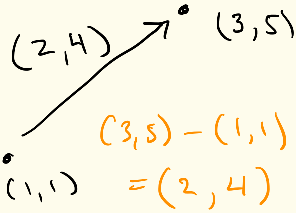
How to make sure: write your solutions by yourself, with your books and notes closed and your devices off.

Do not ever pass off someone else's understanding as your own. It is obvious when you do, and an insult to everyone in the class. I will report every instance. Unsure ⇒ ask, cite.

Vectors describe motion in straight lines.
(linear motion).

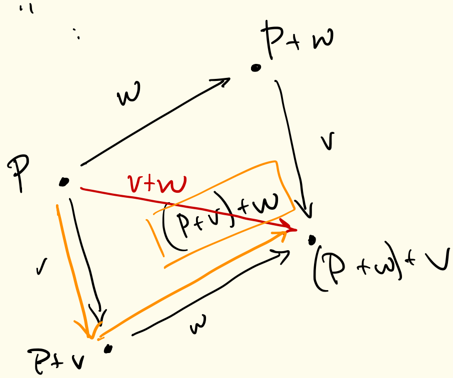
Points describe position. A point is not a vector, but the difference between 2 points is a linear motion (vector)





If v is a vector and w is a vector then v+w is the equivalent linear motion to "v then w":

or "w then v":



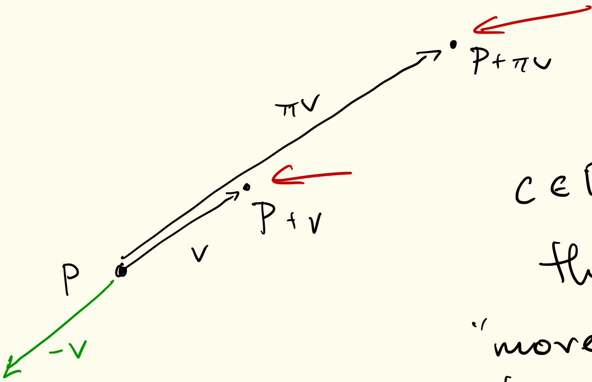
$v + w = w + v$

<u>Question</u> Describe the halfway point from P to Q using the vector $v = Q - P$.

<u>Ans</u> $P + \frac{1}{2} v$.

$Q - \frac{1}{2} v$

If v is a vector and c is a scalar then cv is the linear motion "go c times as far as v in the same direction as c"



$\pi \in \mathbb{R}$.

$c \in \mathbb{R}$ and v vector then cv is a vector "move in dir. v, c times as far as v"

<u>Question</u> What does $\{P + tv : t \in \mathbb{R}\}$ look like?

<u>Ans</u> The line of slope $\bar{v}$ through P.

This is called a parameterization of this line.

meaning: can get all points on the line by substituting values for the parameter t.

<u>Example</u> Find a parameterization for the line in $\mathbb{R}^2$ through the points $(3,7)$ and $(-1,3)$.

<u>Question</u> What would a parameterization of a plane look like? P a point on plane, $\vec{u}, \vec{v}$ = vectors in $\mathbb{R}^3$

$$\{ P + t\vec{u} + s\vec{v} : t, s \in \mathbb{R} \}$$

Linear equations

The field of scalars. A field is a "number system" where you can add, subtract, multiply, and divide (by nonzero elements)

Examples

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (p prime)

$\mathbb{R}(t)$, ...

Non-examples

$\mathbb{Z}$, $\mathbb{R}[t]$, $\mathbb{Z}/n\mathbb{Z}$ (n not prime)

$\mathbb{N}$, ...

For most of this class, can assume
scalar means real number
but will need complex numbers
when we talk about eigenvectors and
eigenvalues. Everything we do will be
valid for any field of scalars.

(homogeneous)
A linear expression over the field of scalars

F is an expression

$$a_1x_1 + \cdots + a_nx_n \qquad \text{where } a_i \in F$$

and x_i are variables.

An affine linear expression is (scalar) + (linear expr.)

A (homogeneous) linear equation is

$$(\text{linear expression}) = 0$$

e.g. $a_1x_1 + \cdots + a_nx_n = 0$

e.g. $3x_1 + 7x_2 = 0$

$$4x + 9y - z = 0$$

An inhomogeneous or affine linear equation is

$$(\text{linear expression}) = (\text{scalar})$$

e.g. $a_1x_1 + \cdots + a_nx_n = c$ $\qquad a_i \in F,\ c \in F,\ x_i$ vars.

e.g., $3x_1 + 7x_2 = 1$

$$4x + 9y - z = 0.$$

Example

Find an affine linear equation for the line through the points $(3,7)$ and $(-1,3)$ in $\mathbb{R}^2$.

Equation will be $ax+by=c$ with

$$3a+7b=c$$
$$-a+3b=c$$

want to solve these for a,b,c.

$$4a+4b=0$$
$$a+b=0 \qquad a=-b$$

e.g., $a=1$, $b=-1$

then $c = 3-7 = -4$
$= -1-3 = -4$

so an equation is $x-y=-4$

<u>Example</u> Find a parameterization of the same line.

[A parameterization is a linear expression