Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. Define *eigenvector*, *eigenvalue*, and *eigenspace*.

Question 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix

$$A = \begin{pmatrix} 2 & 3 & -3 \\ 3 & 5 & -6 \\ 3 & 3 & -4 \end{pmatrix}$$

(a) Find the eigenvalues of T.

(b) Find the eigenvectors of T.

(c) Explain why there is no basis β of $\mathbb{R}^3$ such that $[T]_\beta^\beta$ is *diagonal*.

(d) Find a basis β of $\mathbb{R}^3$ such that $[T]_\beta^\beta$ is *upper triangular*.

(e) Compute A^{100}.

Question 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates a vector counter-clockwise around the origin, by an angle θ.

(a) Find the matrix $[T]_\alpha^\alpha$ for T in the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$.

(b) For what values of θ is T diagonalizable *over the real numbers*?

(c) Find the complex eigenvalues of T. Your answer will depend on θ.

(d) Find the complex eigenvectors of T.

Question 4. Suppose that $f, g : V \to V$ are linear transformations such that $f \circ g = g \circ f$. Suppose that there are bases α and β of V such that $[f]_\alpha^\alpha$ and $[g]_\beta^\beta$ are diagonal. Prove that there is a basis γ of V such that *both* $[f]_\gamma^\gamma$ and $[g]_\gamma^\gamma$ are diagonal.

Question 5. Let V and W be vector spaces. The product vector space is

$$V \times W = \{(v, w) \; : \; v \in V \text{ and } w \in W\}$$

with the following operations:

(i) $\mathbf{0}_{V\times W} = (\mathbf{0}_V, \mathbf{0}_W)$;

(ii) for all $\lambda \in \mathbb{F}$ and all $v \in V$ and all $w \in W$, we have $\lambda \underset{V\times W}{\cdot} (v, w) = (\lambda \underset{V}{\cdot} v, \lambda \underset{W}{\cdot} w)$;

(iii) for all $v, v' \in V$ and $w, w' \in W$, we have $(v, w) \underset{V\times W}{+} (v', w') = (v \underset{V}{+} v', w \underset{W}{+} w')$.

(a) Prove that $V \times W$ is a vector space. You don't have to do every detail: if some of the steps have similar proofs, you can just give an idea of how to complete them; however, don't forget to verify that the operations are well-defined.

(b) Suppose that U is a vector space containing subspaces V and W such that $V \cap W = \{\mathbf{0}\}$ and $V + W = U$. Prove that U is isomorphic to $V \times W$. (Hint: define a function $f : V \times W \to U$ by the formula $f(v, w) = v + w$ and verify that this is an isomorphism.)

(c) Suppose that U is a vector space of dimension n and that $T : U \to U$ is a linear transformation with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that U is isomorphic to $\mathbf{E}_{\lambda_1} \times \cdots \times \mathbf{E}_{\lambda_n}$. (Suggestion: the $n = 2$ case shows most of the important features of this problem, so just focus on that if the general case is confusing.)