Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. (i) Give a formula for, or a procedure for calculating, the determinant of an $n \times n$ square matrix A.

(ii) Define an alternating m-linear function from a vector space V to a vector space W.

Question 2. (i) Compute the determinant of the following matrix:

$$A = \begin{pmatrix} a & 1 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 2 & -3 & -3 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

(ii) For which real numbers a is this matrix invertible?

(iii) When A is not invertible, what is its rank?

Question 3. Let $A \in M_{n\times n}(\mathbb{F})$ be a *nilpotent* matrix. This means that there is some k such that $A^k = 0$. Prove that $\det(A) = 0$.

Question 4. Suppose that $n \geq 2$. Prove the following formula for the determinant of $A \in M_{n\times n}(\mathbb{F})$:

$$\det(A) = \sum_{1\leq i<j\leq n} (-1)^{i+j-1} \det(A_{i,j;1,2}) \det(\tilde{A}_{i,j;1,2})$$

where $A_{i,j;k,\ell} = \begin{pmatrix} A_{i,k} & A_{i,\ell} \\ A_{j,k} & A_{j,\ell} \end{pmatrix}$ and $\tilde{A}_{i,j;k,\ell}$ is the matrix obtained from A by deleting the i-th and j-th rows and the k-th and ℓ-th columns. (Hints: compare the formula to what you get from column expansion; another approach is to show that the formula is alternating and multilinear.)

Question 5. Suppose that V is a vector space of dimension n over the field $\mathbb{F}$. Prove that the dimension of the space of alternating m-linear functions from V to $\mathbb{F}$ is $\binom{n}{m}$.