Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. (i) Suppose that V is a vector space. Define the dual vector space of V.

(ii) Suppose that β is a basis of a vector space V. Define the dual basis of β.

Question 2. The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

are a basis for $\mathbb{R}^3$. Find the dual basis.

Question 3. Give an example of an infinite dimensional vector space V with a basis β and an element $f \in V^*$ that is not in the span of β^*. Make sure to explain how you know f is not in the span of β^*. (Hint: it is possible to find an example using *any* basis of *any* infinite dimensional vector space.)

Question 4. Let $P_n(\mathbb{R})$ be the vector space of polynomials of degree $\leq n$ with coefficients in $\mathbb{R}$. Let c be any element of $\mathbb{R}$. For each $i = 0, \ldots, n$, define a function $D_i : P_n(\mathbb{R}) \to \mathbb{R}$ by the formula

$$D_i(f) = f^{(i)}(c)$$

where $f^{(i)}$ is the i-th derivative of f (by convention, $f^{(0)} = f$).

(i) Prove that all of the functions D_i are linear. (Hint: use induction.)

(ii) Prove that $\delta = \{D_0, \ldots, D_n\}$ is a basis for the dual vector space of $P_n(\mathbb{R})$.

(iii) Find a basis β of $P_n(\mathbb{R})$ such that δ is the dual basis of β.

Question 5. Let $T : V \to W$ be a linear transformation.

(i) Prove that $\ker(T)^* \simeq V^*/T^*(W^*)$. (Hint: construct a function from V^* to $\ker(T)^*$ and show that its kernel is $T^*(W^*)$.)

(ii) Prove that $(W/T(V))^* \simeq \ker(T^*)$. (Hint: define a function $\varphi : (W/T(V))^* \to \ker(T^*)$ by $\varphi(h)(w) = h(w + T(V))$.)