Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. Suppose that $T : V \to W$ is a linear transformation and that α is a basis for V and β is a basis for W. Carefully define the matrix $[T]_\alpha^\beta$ of T with respect to these bases.

Question 2. Let A be the following 4×4 matrix:

$$A = \begin{pmatrix} 2 & -1 & 3 & -2 \\ 4 & -2 & 4 & -2 \\ -2 & 1 & -4 & 3 \\ 2 & -1 & 1 & 0 \end{pmatrix}$$

(i) Find the rank of L_A.

(ii) Find a basis for the image of L_A.

(iii) Find a basis for the kernel of L_A.

(iv) Find a matrix B such that $\mathrm{im}(L_A) = \ker(L_B)$.

(v) Find a matrix C such that $\ker(L_A) = \mathrm{im}(L_C)$.

(vi) Find invertible matrices X and Y such that XAY has the form $\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right)$ where I_k is the $k \times k$ identity matrix (for some k) and 0 indicates a matrix whose entries are all 0.

Question 3. Suppose that X is an $m \times n$ matrix, that A is an $n \times p$ matrix, and that Y is a $p \times q$ matrix. Determine whether the following statements are true or false, and justify your answer:

(i) $\ker(XA) = \ker(A)$ if X is an invertible matrix.

(ii) $\mathrm{im}(XA) = \mathrm{im}(A)$ if X is an invertible matrix.

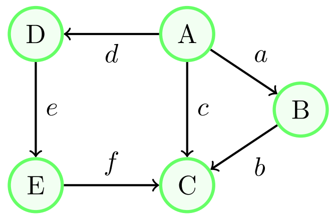
(iii) $\mathrm{rank}(A) = \mathrm{rank}(XAY)$ if X has a left inverse and Y has a right inverse.

Question 4. Let $P_n(\mathbb{R})$ be the vector space of polynomials of degree $\leq n$ with coefficients in $\mathbb{R}$. Define $T : P_2(\mathbb{R}) \to P_4(\mathbb{R})$ by following formula:

$$T(f) = f(x^2) - x^2 f(x) - f(x)$$

Find a basis for the kernel of T and a basis for the image of T.

Question 5. Let R be the set of edges of the following graph and let S be the set of vertices:



For each arrow x in the graph, we write $s(x)$ for the source of x and $t(x)$ for the target of x. For example $s(a) = A$ and $t(a) = B$.

Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$ such that S as a basis of V, and let W be a vector space over $\mathbb{F}$ that has R as a basis.

(i) Show that there is a linear transformation $\partial : W \to V$ such that $\partial(x) = t(x) - s(x)$ for every edge $x \in R$.

(ii) Compute $\operatorname{rank} \partial$.

(iii) Find a basis for the kernel of ∂.

(iv) Find a basis for $V/\operatorname{im}\partial$.

(v) Can you attribute a geometric significance to $\dim \ker \partial$ and $\dim V/\operatorname{im}\partial$? (This is deliberately open ended.)