Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. State the rank-nullity theorem precisely.

Question 2. Let $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation $T(f) = (f(0), f'(1) + f(2), f'(0))$. Find bases for the kernel of T and for the image of T.

Question 3. Suppose that V and W are vector spaces of dimensions 6 and 4, respectively.

(a) Suppose that $T : V \to W$ is a linear transformation. What are the possibilities for $\dim \ker(T)$ and $\dim \operatorname{im}(T)$?

(b) Suppose that $T : W \to V$ is a linear transformation. What are the possibilities for $\dim \ker(T)$ and $\dim \operatorname{im}(T)$?

(c) What are the possibilities for the dimension of the space of solutions to a system of n homogeneous linear equations in m unknowns? Hint: you may have different answers when $m \geq n$ and when $m \leq n$. (Note: a homogeneous linear equation is one that looks like $a_1x_1 + \cdots + a_nx_n = 0$ for some $a_1, \ldots, a_n \in \mathbb{F}$.)

Question 4. Let F be a finite field with q elements. Let $V = P(F)$ and let $W = \operatorname{Func}(F, F)$ be the vector space of all functions from F to F.

(i) Compute $\dim V$.

(ii) Compute $\dim W$.

(iii) Let $T : V \to W$ be the transformation that sends a polynomial $f(x)$ to the function it represents. Prove that T is a linear transformation.

(iv) Prove that there are infinitely many polynomials $f \in V$ that represent the zero function.

Question 5. Let V be a vector space over F containing a subspace U. A *coset* of U in V is a subset of the form

$$\mathbf{v} + U = \{\mathbf{v} + \mathbf{u} \; : \; \mathbf{u} \in U\}.$$

Let V/U be the set of cosets of U in V. Define operations and a zero element for V/U:

(i) The zero element is the coset $\mathbf{0} + U = U$.

(ii) The addition law is $(\mathbf{v} + U) + (\mathbf{w} + U) = (\mathbf{v} + \mathbf{w}) + U$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.

(iii) The scalar multiplication law is $\lambda.(\mathbf{v} + U) = (\lambda.\mathbf{v}) + U$ for all $\mathbf{v}$ in V and all λ in F.

Do all of the following:

(a) Prove that V/U is a vector space with these operations.

(b) Show that the function $\pi : V \to V/U$, given by $\pi(\mathbf{v}) = \mathbf{v}+U$, is a linear transformation.

(c) Show that $\ker(\pi) = U$.