Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. Explain what a basis of a vector space is. You do not have to define a vector space or field, but you should explain all other linear-algebraic terms you use.

Question 2. Let

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \,:\, x + y + z + w = 0\}.$$

(i) Prove that the vectors $(1, -1, 0, 0)$, $(0, 1, -1, 0)$, and $(0, 0, 1, -1)$ form a basis for W.

(ii) Find the coordinates of the vector $(-4, 2, 3, -1)$ in this basis.

Question 3. Let $P_n(F)$ be the vector space of polynomials of degree $\leq n$ with coefficients in F. Let a be an element of F and let

$$V = \{f \in P_n(F) \,:\, f(a) = 0\}.$$

Find a basis for V and use this to compute the dimension of V. (Hint: it may help to think about the special case where $a = 0$ first.)

Question 4. Suppose that U and V are subspaces of a vector space W. Prove that

$$\dim(U + V) + \dim(U \cap V) = \dim U + \dim V.$$

(Hint: start with a basis of $U \cap V$ and extend it to bases of U and of V. Then prove that the union of these bases is a basis for $U + V$. Use inclusion.)

Question 5. Let V be a vector space and let $S \subseteq V$ be a *minimal* generating subset. This means that $\mathrm{span}(S) = V$ but if $\mathbf{v} \in S$ then $\mathrm{span}(S - \{\mathbf{v}\}) \neq V$. Prove that S is a basis of V.