Instructions: Work alone. You may consult any references you like while thinking about this assignment, but when actually writing up your solutions for submission, you should not consult sources other than your own brain.

Question 1. State the definition of a vector space. Your definition need not be an exact reproduction of the definition in the textbook, but it should be equivalent to the definition in the textbook.

Question 2. Suppose that V is a vector space over a field F. Assume that $\mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$. Prove rigorously that

$$(a+b)(\mathbf{v}+\mathbf{w}) = a\mathbf{v} + a\mathbf{w} + b\mathbf{v} + b\mathbf{w}.$$

Question 3. Determine which of the following are vector subspaces. Your answer should be 'Yes', 'No', or 'It depends', with a brief explanation.

(i) Let a, b, and c be real numbers. Let

$$W = \{(x,y,z) \in \mathbb{R}^3 \;:\; ax+by+cz=0\}.$$

Is W a vector subspace of $\mathbb{R}^3$?

(ii) Suppose that V is a vector space containing vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Let

$$W = \{(a,b) \in F^2 \;:\; a\mathbf{u} + b\mathbf{v} = \mathbf{w}\}.$$

Is W a vector subpace of F^2?

Question 4. Suppose that U is a vector space over a field F containing subspaces V and W. Let

$$V + W = \{\mathbf{v} + \mathbf{w} \;:\; \mathbf{v} \in V \text{ and } \mathbf{w} \in W\}.$$

Prove that $V+W$ is a subspace of U.

Question 5. Suppose that U is a vector space over a field F and S is a subset of U. Define:

$$A = \bigcap_{\substack{V \supseteq S \\ V \text{ is a sub-} \\ \text{space of } U}} V = \{\mathbf{v} \in U \;:\; \mathbf{v} \in V \text{ for every subspace } V \subseteq U \text{ that contains } S\}$$

$$B = \{a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n \;:\; a_1, \ldots, a_n \in F \text{ and } \mathbf{x}_1, \ldots, \mathbf{x}_n \in S\}$$

Prove that $A = B$ and that both are subspaces of U.