Definiton 1. A *permutation* of $\{1, \dots, n\}$ is a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. A permutation is called an *adjacent transposition* if there is an i such that $\sigma(i) = i+1$ and $\sigma(i+1) = i$ and $\sigma(j) = j$ for all $j \neq i$. A permutation is called a *transpotion* if there are $i \neq j$ such that $\sigma(i) = j$ and $\sigma(j) = i$ and $\sigma(k) = k$ for all $k \neq i, j$.

Definiton 2. The *permutation matrix* of a permutation of $\{1, \dots, n\}$ is the $n \times n$ matrix A^σ such that $A^\sigma_{ij} = 0$ unless $i = \sigma(j)$.

Question 1. (i) Let $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ be the permutation with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 4$, and $\sigma(4) = 1$. Write down the permutation matrix A^σ of σ.

(ii) Describe the shape of the permutation matrix of transposition and of an adjacent transposition.

Question 2. (i) Suppse that σ is a permutation of $\{1, \dots, n\}$. Prove that $A^\sigma(\mathbf{e}_j) = \mathbf{e}_{\sigma(j)}$.

(ii) Suppose that σ and τ are permutations of $\{1, \dots, n\}$. Prove that $A^\sigma A^\tau = A^{\sigma \circ \tau}$.

Definiton 3. The *sign* of a permutation σ is the following product:

$$\operatorname{sgn}(\sigma) = \prod_{\substack{0<i<j\leq n \\ \sigma(i)>\sigma(j)}} (-1)$$

Question 3. Compute the sign of the permutations σ from Question 1. Show that the sign of a transposition is always -1.

Question 4. Show that every permutation is a composition of adjacent transpositions. (Hint: show that every permutation is a composition of transpositions, and that every transposition is a composition of adjacent transpositions.) [Feel free to skip this one—it's not really linear algebra.]

Question 5. Let $D : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be an alternating linear function. Prove that $D(A^\sigma) = \operatorname{sgn}(\sigma) D(I_n)$ where I_n is the $n \times n$ identity matrix. (Hint: write σ as a product of adjacent transpositions $\tau_1 \cdots \tau_m$. Show that $D(A^{\tau_1 \cdots \tau_m}) = -D(A^{\tau_1 \cdots \tau_{m-1}})$.)

Definiton 4. Define $\gamma_n : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ by the following formula:

$$\gamma_n(A) = \sum_\sigma \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

Question 6. Suppose that A' is obtained from A by exchanging two adjacent columns. Show that $\gamma_n(A') = -\gamma_n(A)$. (Suggestion: work out what happens when A is a 2×2 matrix or a 3×3 matrix to get an idea of what is going on.)

Question 7. Suppose that

$$\begin{aligned} A &= \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{v}_i + \mathbf{v}_i' & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \end{pmatrix} \\ B &= \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{v}_i & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \end{pmatrix} \\ C &= \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{v}_i' & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \end{pmatrix} \end{aligned}$$

and prove that $\gamma_n(A) = \gamma_n(B) + \gamma_n(C)$. (Again, it might help to think about the 2×2 or 3×3 case if it isn't clear how to proceed in general.)