Lemma 1 (Exchange lemma)**.** *Suppose that $T \subseteq S$ are subsets of a vector space V. If $\mathbf{v} \in \operatorname{span}(S) - \operatorname{span}(T)$ then there is some $\mathbf{w} \in S$ such that $(S - \{\mathbf{w}\}) \cup \{\mathbf{v}\}$ has the same span as S.*

Proof. Since $\mathbf{v} \in \operatorname{span}(S)$ there are some vectors $\mathbf{w}_j \in S$ and coefficients $a_j \in F$ such that

$$\mathbf{v} = \sum a_j \mathbf{w}_j.$$

Since $\mathbf{v} \notin \operatorname{span}(T)$, there is at least one $\mathbf{w}_j$ that is not in T and for which $a_j \neq 0$. Then

$$\mathbf{w}_j = a_j^{-1}\mathbf{v} - \sum_{i \neq j} a_j^{-1} a_i \mathbf{w}_i$$

and this means that, if we define $\mathbf{w} = \mathbf{w}_j$, that

$$\operatorname{span}\big((S - \{\mathbf{w}\}) \cup \{\mathbf{v}\}\big) = \operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span}(S)$$

since $\mathbf{w} \in \operatorname{span}(S \cup \{\mathbf{v}\})$ and $\mathbf{v} \in \operatorname{span}(S)$. □

Theorem 1 (Replacement theorem)**.** *Suppose that V is a vector space, that $S \subseteq V$ is a set of linearly independent vectors, and that $T \subseteq V$ is a generating set of vectors. Then the cardinality of S is $\leq$ the cardinality of T.*

Proof. The strategy of the proof is to replace T with another spanning collection of vectors T' having the same cardinality but containing S. We do this by induction on the size of S.

STEP 1. Choose an ordering on the elements of S. If S is infinite, this should be a *well*-ordering. In the finite case, this means we can write $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

STEP 2. For each $i = 0, \ldots, n$, we will inductively define a set T_i of vectors of V such that $\mathbf{v}_j \in T_i$ for all $j \leq i$ and $\operatorname{span}(T_i) = \operatorname{span}(T)$. We begin with $T_0 = T$.

STEP 3. Assume, by induction, that $\operatorname{span}(T_i) = \operatorname{span}(T)$ and $\mathbf{v}_1, \ldots, \mathbf{v}_i \in T_i$. By the exchange lemma, we can find a $\mathbf{w} \in T - \{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ such that $\operatorname{span}\big((T_i - \{\mathbf{w}\}) \cup \{\mathbf{v}_{i+1}\}\big) = \operatorname{span}(T_i)$. Define $T_{i+1} = (T_i - \{\mathbf{w}\}) \cup \{\mathbf{v}_{i+1}\}$.

□