In all of the following questions, F will be a field and P will be the vector space of polynomials with coefficients in F. We will also have:

$$Q = \{f \in F \,:\, f(s) = f(-s) \text{ for every } s \in F\}$$

And we write $P_n \subseteq P$ for the subspace of polynomials of degree $\leq n$.

Question 1. Let F be a field in which $2 = 0$ (for example, F could be $\mathbb{Z}/2\mathbb{Z}$, but this is not the only field where $2 = 0$). Prove that $Q = P$.

Question 2. Let F be the field $\mathbb{Z}/3\mathbb{Z}$.

(i) Prove that $s^3 = s$ for every $s \in F$. Deduce that $x^3 - x$ is an even function.

(ii) Prove that, for every $f \in P$, there is some $g \in P_2$ such that f and g define the same function on F. Does this mean that f and g are the same polynomial? (You may want to look up what it means for two polynomials to be the same.)

(iii) Prove that $\{1, x^2\}$ is a basis for $Q \cap P_2$.

(iv) Prove that

$$\{1, x^2, x^4, x^6, x^8, \ldots\} \cup \{x^3 - x, x^5 - x^3, x^7 - x^5, \ldots\}$$

is a basis for Q.

Question 3. Find a basis for Q when F is the field $\mathbb{Z}/5\mathbb{Z}$. Hint: think about the polynomial $x^5 - x$.

Question 4. Prove that if F is any finite field then there is a nonzero polynomial f such that $f(s) = 0$ for all $s \in F$.

Question 5. Suppose that F is an infinite field and $2 \neq 0$ in F. Prove that

$$\{1, x^2, x^4, x^6, x^8, \ldots\}$$

is a basis for Q.

Question 6. Find a basis for Q when F is any finite field. (This problem is really beyond the scope of this course, but you may be able to do it when $F = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number.)