The vector space axioms

Math 3135–001, Spring 2017

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Definition 1. A *vector space* over a field F is a set V, equipped with

- an element $\mathbf{0} \in V$ called *zero*,
- an *addition* law $\alpha : V \times V \to V$ (usually written $\alpha(\mathbf{v}, \mathbf{w}) = \mathbf{v} + \mathbf{w}$), and
- a *scalar multiplication* law $\mu : F \times V \to V$ (usually written $\mu(\lambda, \mathbf{v}) = \lambda.\mathbf{v}$)

satisfying the following axioms:

VS1 (commutativity of vector addition) For all $\mathbf{v}$ and $\mathbf{w}$ in V, we have $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

VS2 (associativity of vector addition) For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, we have $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

VS3 ($\mathbf{0}$ is a left identity for vector addition) For all $\mathbf{v} \in V$ we have $\mathbf{0} + \mathbf{v} = \mathbf{v}$.

VS4 (existence of additive inverses) For each $\mathbf{v} \in V$ there is some $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

VS5 (1 is an identity for scalar multiplication) For all $\mathbf{v} \in V$ we have $1.\mathbf{v} = \mathbf{v}$.

VS6 (associativity of scalar multiplication) For all $\lambda, \mu \in F$ and all $\mathbf{v} \in V$, we have $(\lambda\mu).\mathbf{v} = \lambda.(\mu.\mathbf{v})$.

VS7 (distributivity of scalar multiplication over vector addition) For all $\lambda \in F$ and all $\mathbf{v}$ and $\mathbf{w}$ in V, we have $\lambda.(\mathbf{v} + \mathbf{w}) = \lambda.\mathbf{v} + \lambda.\mathbf{w}$.

VS8 (distributivity of scalar multiplication over scalar addition) For all $\lambda, \mu \in F$ and all $\mathbf{v} \in V$ we have $(\lambda + \mu).\mathbf{v} = \lambda.\mathbf{v} + \mu.\mathbf{v}$.

Lemma 2. *If V satisfies* ______________ *and $\mathbf{e} \in V$ is a vector such that $\mathbf{0} + \mathbf{e} = \mathbf{0}$ then $\mathbf{e} = \mathbf{0}$.*

Proof. Then we have:

$$\begin{aligned} \mathbf{0} &= \mathbf{0} + \mathbf{e} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= \mathbf{e} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \end{aligned}$$

□

Lemma 3. *If V satisfies* ______________ *then $0.\mathbf{0} = \mathbf{0}$.*

Proof. We have:

$$\begin{aligned} \mathbf{0} + 0.\mathbf{0} &= 1.\mathbf{0} + 0.\mathbf{0} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= (1+0).\mathbf{0} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 1.\mathbf{0} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= \mathbf{0} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \end{aligned}$$

Therefore $0.\mathbf{0} = \mathbf{0}$, by ______________. □

Lemma 4. *If V satisfies* ______________ *and if $\mathbf{v} \in V$ is any element then $0.\mathbf{v} = \mathbf{0}$.*

Proof. We have:

$$\begin{aligned} \mathbf{0} + 0.\mathbf{v} &= 0.\mathbf{0} + 0.\mathbf{v} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 0.(\mathbf{0} + \mathbf{v}) && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 0.\mathbf{v} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_.} \end{aligned}$$

Therefore $0.\mathbf{v} = \mathbf{0}$ by ______________. □

Theorem 5. *If V satisfies* ______________ *and $\mathbf{v} \in V$ then $\mathbf{v} + (-1).\mathbf{v} = \mathbf{0}$.*

Proof. We have:

$$\begin{aligned} \mathbf{v} + (-1).\mathbf{v} &= 1.\mathbf{v} + (-1).\mathbf{v} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= (1 + (-1)).\mathbf{v} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 0.\mathbf{v} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= \mathbf{0} && \text{by \_\_\_\_\_\_\_\_\_\_\_\_\_\_.} \end{aligned}$$

□

Corollary 6. *If V satsifies Axioms* ____________________________ *then V is a vector space.*

Lemma 7. *If V satisfies VS2, VS3, VS5, and VS8 then, for all $\mathbf{v} \in V$, we have $\mathbf{v} + \mathbf{0} = \mathbf{v}$.*

[Hint: $\mathbf{v} + \mathbf{0} = (1 + 0)\mathbf{v}$ and $1 + 0 = 0 + 1$; thanks to Elliot for this suggestion!]

Theorem 8. *If V satisfies VS2, VS3, VS5, VS7, and VS8 then V satisfies VS1.*

[Hint: expand $(-1).v+(1+1).(\mathbf{v}+\mathbf{w})+(-1).\mathbf{w}$ in two ways, either using VS7 before VS8 or VS8 before VS7.]