Math 3110: Number Theory

Exploration 4: Quadratic forms

April 11, 2016

1 Change of variables

Definition 1. A *binary integer quadratic form* is a function $q : \mathbb{Z}^2 \to \mathbb{Z}$ of the form

$$q(x, y) = ax^2 + bxy + cy^2 \tag{Q}$$

where a, b, and c are integers.

The expression in (Q) can be rewritten using vector notation:

$$q(x\mathbf{e}_1 + y\mathbf{e}_2) = ax^2 + bxy + cy^2$$

We could try to reexpress q in terms of any basis. Let's recall the definition:

Definition 2. A *basis* of $\mathbb{Z}^2$ is a pair of vectors, $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ such that every vector in $\mathbb{Z}^2$ is an integer linear combination of $\mathbf{u}$ and $\mathbf{v}$.

In other words, for $\mathbf{u}$ and $\mathbf{v}$ to be a basis means that for every vector $\mathbf{w} \in \mathbb{Z}^2$, it is possible to find some integers x and y such that

$$\mathbf{w} = x\mathbf{u} + y\mathbf{v}.$$

Suppose that $\mathbf{u}$ and $\mathbf{v}$ form a basis for $\mathbb{Z}^2$. We want to find a formula for

$$q(x\mathbf{u} + y\mathbf{v}).$$

Question 3. Suppose that $q(x\mathbf{e}_1 + y\mathbf{e}_2) = ax^2 + bxy + cy^2$. Explain why there must be some integers a', b', and c' such that

$$q(x\mathbf{u} + y\mathbf{v}) = a'x^2 + b'xy + c'y^2. \tag{Q'}$$

Thus reexpressing q in another basis is the same as solving for a', b', and c'.

Question 4. Knowing that $q(x\mathbf{u} + y\mathbf{v}) = a'x^2 + b'xy + c'y^2$, compute

$$\begin{array}{ll} q(\mathbf{u}) & q(\mathbf{u} + \mathbf{v}) \\ q(\mathbf{v}) & q(\mathbf{u} - \mathbf{v}) \end{array}$$

in terms of a', b', and c'. Use these to get formulas for a', b', and c' in terms of special values of q.

2 Conway's topograph

We can see the values of a quadratic form by labelling a topograph. Remember that vector $\mathbf{u}$ and $\mathbf{v}$ appear on opposite sides of an edge of the topograph if and only if $\mathbf{u}$ and $\mathbf{v}$ form a basis for the topograph. We are going to decorate the topograph in such a way that we can easily read off the quadratic form in any given basis from the labelling.

We have just seen that if

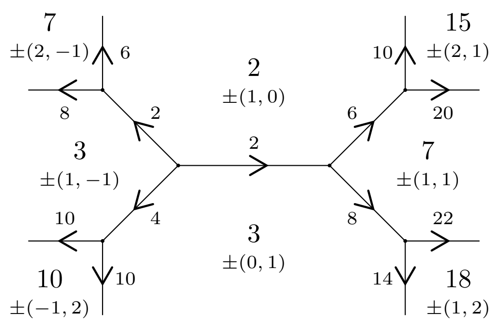
$$q(x\mathbf{u} + y\mathbf{v}) = ax^2 + bxy + cy^2$$

then

$$\begin{aligned} a &= q(\pm\mathbf{u})b & &= q(\pm(\mathbf{u}+\mathbf{v})) - q(\pm\mathbf{u}) - q(\pm\mathbf{v}) \\ &= q(\pm\mathbf{u}) + q(\pm\mathbf{v}) - q(\pm(\mathbf{u}-\mathbf{v})) \\ &= \frac{q(\pm(\mathbf{u}+\mathbf{v})) - q(\pm(\mathbf{u}-\mathbf{v}))}{2} \\ c &= q(\pm\mathbf{v}). \end{aligned}$$

We will therefore label the region $\mathbf{u}$ with the value $a = q(\mathbf{u})$, the region $\mathbf{v}$ with the value $c = q(\mathbf{v})$, and the edge between $\mathbf{u}$ and $\mathbf{v}$ with the value b. However, we need to put an arrow on the edge so that we know which way we have computed b.

Here is a picture of the topograph for $q(x, y) = 2x^2 + 2xy + 3y^2$:



Question 5. What patterns can you observe in the topograph? Can you prove that any of these hold more generally?

3 The discriminant