10/9/19

Notes for class coverage.

Goals: defns of log scheme, maps of log schemes, coherent log structures, some intuition

References: "Log structures of Fontaine-Illusie" – Kazuya Kato ← too short

"Lectures on logarithmic geometry" – Arthur Ogus ← too long

Skip for time

~~Recall: A monoid $(M, \cdot)$ is a set M and an operation $\cdot$ so that $\cdot$ is associative and has an identity 1_M. Our monoids will be assumed commutative.~~

~~A morphism of monoids $\varphi: M \to N$ is a function with $\varphi(m \cdot m') = \varphi(m) \cdot \varphi(m')$ $\forall m, m' \in M$ and $\varphi(1_M) = 1_N$.~~

~~These are good for things you know how to multiply, but not necessarily to divide.~~

Review question: What is an fpqc sheaf on a scheme S?

... (Betting we need a reminder) ...

probably skip for time.

Defn: A sieve $\mathcal{R}$ on Sch/S is a subcategory so that whenever the red arrows in

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z \\ & Y \arrow[u] \end{tikzcd}$$

are in $\mathcal{R}$, so is the black arrow.

If $\{U_i \to S\}$ is some collection of arrows, the sieve generated by the $\{U_i\}$ is the subcategory of Sch/S containing precisely the arrows $\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$ factoring through some $U_i \to S$: $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed] & Y \arrow[dl, dashed] \arrow[d] \\ U_i \arrow[r] & S \end{tikzcd}$

Given an fpqc cover of S, i.e. an affine open cover $\{T_i\}$ of S and affine, flat, surjective maps $\{U_i \to T_i\}$, the associated fpqc covering sieve is the sieve generated by $\{U_i \to S\}$.

A functor $\mathcal{F}: (Sch/S)^{op} \to Set$ is an fpqc sheaf if, for all $U \to S$ and fpqc covering sieves $\mathcal{R}$ of U,

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\mathcal{R}) = \left\{ \begin{tikzcd} & & \mathcal{F} \arrow[d] \\ \mathcal{R} \arrow[r] \arrow[urr, dashed] & Sch/U \arrow[r] & Sch/S \end{tikzcd} \right\}$$

$$= \lim_{V \to U \in \mathcal{R}} \mathcal{F}(V \to U)$$

is an isomorphism.

An *étale sheaf* is the same thing, considering the sieves $\mathcal{R}$ generated by the sets of arrows $\{U_i \to U\}$ which are étale and jointly surjective. This is weaker than fpqc since étale maps are flat, qcpt.

Example: The structure sheaf of X may be considered as an étale sheaf on Sch/X by

$$\mathcal{O}_X(U \xrightarrow{f} X) = \Gamma(U, f^*\mathcal{O}_X) = \Gamma(U, \mathcal{O}_U).$$

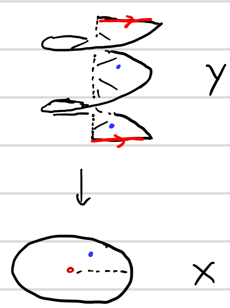
Similarly, we can define a sheaf $\mathcal{O}_X^*$ by

$$\mathcal{O}_X^*(U \xrightarrow{f} X) = \Gamma(U, \mathcal{O}_U^*).$$

Example: (Étale descent).

Let $f : Y = \operatorname{Spec} k[u^{\pm}] \longrightarrow X = \operatorname{Spec} k[t^{\pm}]$

be defined by $u^2 \longleftarrow t$



f is an étale cover

Exercise:

- The set $\varprojlim \left(\mathcal{O}_X(Y) \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} \mathcal{O}_X(Y \times_X Y) \right)$ can be thought of as "functions on Y whose values are invariant along the fibers of f"
- The sheaf condition says
$$\mathcal{O}_X(X) \to \mathcal{O}_X(\mathcal{R}) \simeq \varprojlim \mathcal{O}_X(Y) \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} \mathcal{O}_X(Y \times_X Y)$$
is an isomorphism. Verify this by an explicit computation.

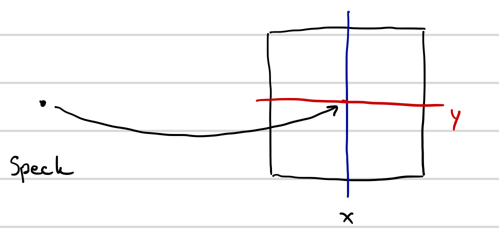
Recall: An effective Cartier divisor on a scheme X is a subscheme D locally cut out by one equation. That is, there is an open cover $\{U_i\}$ of X and $f_i \in \Gamma(U_i, \mathcal{O}_X)$ with $D|_{U_i} = V(f_i)$ so that

i) each f_i is a nonzero divisor (so we're actually cutting things out)

ii) $f_i|_{U_i \cap U_j} = u_{ij} \cdot f_j|_{U_i \cap U_j}$ for some $u_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$, all i,j (the f_is cut out the same thing where they overlap)

This is what "codimension one" is if you think being cut out by one equation is being codimension.
(if you think being codimension one means you are maximal, closed, proper subset, you get Weil divisors)

Problem: Cartier divisors don't behave well under base change:



$\mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$

$0 \longleftarrow\!\shortmid x$

$0 \longleftarrow\!\shortmid y$

Not only do x and y no longer define Cartier divisors, but they also cannot be told apart.

This causes us to experience negative emotions.

The solution offered by log geometry is to decouple the divisors from the ring.

Defn: A log scheme is a pair $(X, \varepsilon : M_X \to \mathcal{O}_X)$ of

i) A scheme X

ii) A morphism of étale sheaves of commutative monoids $\varepsilon : M_X \to \mathcal{O}_X$

so that $\varepsilon : \varepsilon^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism.

Think of M_X = "sheaf of our favorite local equations of Cartier divisors"

$\varepsilon^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*$ is so we can tell algebraically if two local equations cut out the same divisor.

We can also define a quotient sheaf

$$\overline{M}_X = M_X / \varepsilon^{-1}(\mathcal{O}_X^*)$$

$$= \operatorname{colim} \left\{ \varepsilon^{-1}(\mathcal{O}_X^*) \underset{1}{\overset{\text{include}}{\rightrightarrows}} M_X \right\}$$

called the <u>characteristic sheaf</u> of X.

Writing down what a quotient sheaf is:

$$\overline{M}_X(U) = \left\{ (f_i) \in \prod_i \Gamma(U_i, M_X)/\Gamma(U_i, \mathcal{O}_X^*) \;\middle|\; U_i \text{ cover } U,\ f_i|_{U_i \cap U_j} = u_{ij} \cdot f_j|_{U_i \cap U_j} \text{ for some } u_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*) \right\}.$$

= "sheaf of our favorite Cartier divisors"

The thing is, at this point it's rather hard to write a log structure down: we need to specify a monoid for every scheme ever, satisfying the identity-on units condition, and satisfying the sheaf condition for every étale cover ever.

Associated log structures allow us not to worry about getting the units right: just glue them in...

<u>Defn:</u> If X is a scheme and $M_X \xrightarrow{\varepsilon} \mathcal{O}_X$ is any morphism of étale sheaves of monoids, we may take the pushout

$$\begin{tikzcd}
\varepsilon^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d, "\varepsilon"'] & M_X \arrow[d] \arrow[ddr, bend left, "\varepsilon"] & \\
\mathcal{O}_X^* \arrow[r] \arrow[drr, bend right, equal] & M_X^a \arrow[dr, dashed, "\exists!\, \varepsilon^a"] \arrow[ul, phantom, "\lrcorner" very near start] & \\
& & \mathcal{O}_X
\end{tikzcd}$$

Then $M_X^a \xrightarrow{\varepsilon^a} \mathcal{O}_X$ defines an associated log structure on X.

Explicitly, M_X^a is the sheafification of the presheaf

$$\mathcal{F}(U) = \{ (f,u) \in M_X(U) \times \mathcal{O}_X^*(U) \} / \{ (f,0) \sim (0, \varepsilon(f)) \text{ whenever } \varepsilon(f) \in \mathcal{O}_X^* \}.$$

Chaining this together w/ the constant sheaf functor gives a way to construct log structures:

<u>Construction</u>: Suppose M is a monoid. Given any morphism $\varepsilon: M \to \Gamma(X, \mathcal{O}_X)$, there is an associated morphism $\varepsilon: \underline{M} \to \mathcal{O}_X$ from the constant sheaf w/ values M to $\mathcal{O}_X$. Taking $\varepsilon^a: \underline{M}^a \to \mathcal{O}_X$ gives us a log structure.

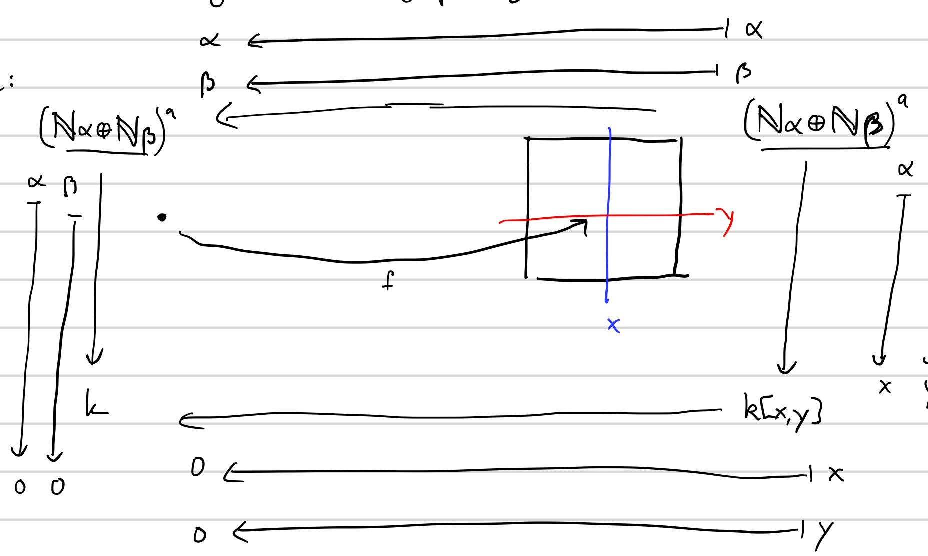
We say $\varepsilon: M \to \Gamma(X, \mathcal{O}_X)$ is a <u>chart</u> for a log structure. (Differs from lit.)

<u>Defn</u>: Let $(X, \varepsilon: M_X \to \mathcal{O}_X)$ be a log scheme. If there is an étale covering $\{U_i \to X\}$ of X so that each $(U_i, \varepsilon: M_X|_{U_i} \to \mathcal{O}_{U_i})$ arises from a chart, we say the log structure on X is <u>quasicoherent</u>.

These are the log structures that sane people think about, in the same way that QCoh sheaves are the $\mathcal{O}_X$-modules that sane people think about.

Now we can build log structures by picking local charts.

<u>Example</u>:

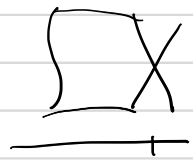


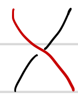
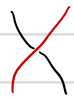
Can still tell α, β apart because they are decoupled from the ring.

Therefore, we may think of log schemes as "schemes w/ a family of favorite divisors, where the divisors possibly extend past the scheme itself."

Preview for Friday:

In a family of curves like we saw on Monday,



The divisors , , and are particularly interesting.

Log structures allow us to remember them even when working only in the special fiber.