Logarithmic geometry, curves, and moduli

Lectures at the Wise course

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Moduli of curves

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> *Working with a non-complete moduli space is like keeping change in a pocket with holes*
>
> Angelo Vistoli

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- $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ - moduli of *stable* curves, a modular compactification.
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- What's so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the *logarithmically smooth* curves.

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Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}_{\underline{X}}^* \to \mathcal{O}_{\underline{X}}^*$ is an isomorphism.

Types of logarithmic structures

- We say that $(\underline{X}, M_X)$ is *coherent* if *étale locally* at every point there is a *finitely generated* monoid P and a local chart $P_X \to \mathcal{O}_{\underline{X}}$ for X.

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- We say that a logarithmic structure is *fine and saturated* (or fs) if it is coherent with local charts $P_X \to \mathcal{O}_X$ with P *integral and saturated*.

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$$\begin{array}{ccccc} & & & & X \\ & & & \overset{\exists}{\nearrow} & \downarrow \\ T_0 & \hookrightarrow & T & \longrightarrow & Y \end{array} \qquad (T_0 \to X)$$

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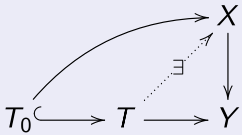

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The morphism is *logarithmically étale* if the lifting in (2) is unique.

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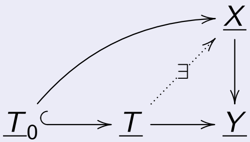

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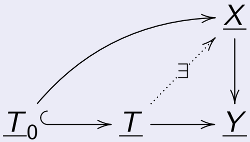

since $\underline{X} \to \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback. ♠

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Then $X \to Y$ *is logarithmically smooth.*
If also the cokernel is finite then $X \to Y$ *is logarithmically étale.*

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Theorem (K. Kato)

X, Y *fine*, $Q_Y \to M_Y$ *a chart.*

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One direction:

$$\begin{array}{ccccc} \underline{X} & \longrightarrow & \underline{Y} \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P] & \longrightarrow & \operatorname{Spec} \mathbb{Z}[P] \\ & & \downarrow & & \downarrow \\ & & \underline{Y} & \longrightarrow & \operatorname{Spec} \mathbb{Z}[Q] \end{array}$$

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- the fibers are curves i.e. pure dimension 1 schemes.

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Moduli of stable log curves

We define a category $\overline{\mathcal{M}}_{g,n}^{\log}$ of stable log curves: objects are log (g,n)-curves $X \to S$ and arrows are fiber diagrams compatible with sections

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There is a forgetful functor

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{\log} & \longrightarrow & \mathfrak{LogSch}^{\mathrm{fs}} \\ (X \to S) & \mapsto & S. \end{array}$$

So $\overline{\mathcal{M}}_{g,n}^{\log}$ is a category fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

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Given a stable curve $\underline{X} \to \underline{S}$ we define

$$X^{\min} = \underline{X} \times_{\overline{\mathcal{M}}_{g,n+1}} \overline{\mathcal{M}}^{\log}_{g,n+1} \qquad \text{and} \qquad S^{\min} = \underline{S} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{\log}_{g,n}.$$

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The logarithmic structures $X^{\min} \to S^{\min}$ are called the *minimal* or *basic* logarithmic structures on a log curve.

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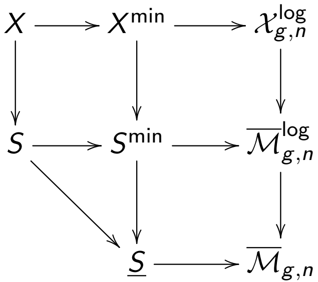
We write

$$S^{\text{min}} = (\underline{S}, M_{X/S}^{S}) \qquad \text{and} \qquad X^{\text{min}} = (\underline{X}, M_{X/S}^{X}).$$

Fundamental diagram



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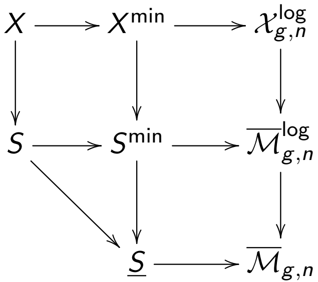

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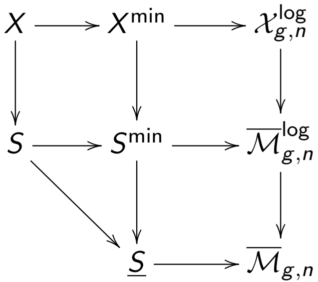

- $\overline{\mathcal{M}}_{g,n}^{\log}$ parametrizes stable log curves over $\mathfrak{LogSch}^{\mathrm{fs}}$
- $\overline{\mathcal{M}}_{g,n}$ parametrizes minimal stable log curves over $\mathfrak{Sch}$.

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Observation

The minimal object exists, with characteristic sheaf dual to the lattice in the corresponding space of tropical curves.