Problem Set #5

Math 2135 Spring 2020

Due Friday, April 17

1. There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the vectors

$$\mathbf{x}^1 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{x}^2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors with the following eigenvalues:

$$\lambda_1 = 7 \qquad \lambda_2 = -1 \qquad \lambda_3 = 5$$

Compute $T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (in the standard basis).

Solution. We have

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2}\mathbf{x}^1 - \mathbf{x}^2 + \frac{1}{2}\mathbf{x}^3$$

so

$$\begin{aligned} T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= -\frac{1}{2}T(\mathbf{x}^1) - T(\mathbf{x}^2) + \frac{1}{2}T(\mathbf{x}^3) \\ &= -\frac{7}{2}\mathbf{x}^1 + \mathbf{x}^2 + \frac{5}{2}\mathbf{x}^3 \\ &= \begin{pmatrix} -10 \\ -1 \\ 7 \end{pmatrix} \end{aligned}$$

□

2. Let P_n be the vector space of polynomials of degree $\leq n$ and let $T : P_n \to P_n$ be the linear transformation $T(p)(x) = xp'(x)$. Find all eigenvalues and eigenvectors of T.

Solution. We wish to solve $xp' = \lambda p$ for λ and p. If $p = a_n x^n + a_{n-1}x^{n-1} + \cdots$ then $xp' = na_n x^n + (n-1)a_{n-1}x^{n-1} + \cdots$. Setting this equal to λp, we get $ma_m = \lambda a_m$ for all m. Therefore λ can be $0, 1, \ldots, n$ and the λ-eigenspace is spanned by x^λ. □

3. Define a sequence by the following recursive formula:

$$\begin{aligned} X_0 &= 0 \\ X_1 &= 1 \\ X_n &= 6X_{n-1} - 9X_{n-2} && \text{for } n \geq 2. \end{aligned}$$

Compute X_{2020}. Make sure to justify your answer with a proof! (Hint: the method we used for the Fibonacci numbers will not work exactly the same way here, but it will get you started. Look for a basis of generalized eigenvectors instead of a basis of eigenvectors.)

Solution. We have

$$\begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The characteristic polynomial of $A = \begin{pmatrix} 6 & -9 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. Therefore 3 is the only eigenvalue. The 3-eigenspace is

$$\ker \begin{pmatrix} 6-3 & -9 \\ 1 & -3 \end{pmatrix} = \ker \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

We have $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. In the basis $U = \begin{pmatrix} \mathbf{u}^1 & \mathbf{u}^2 \end{pmatrix}$ consisting of $\mathbf{u}^1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{u}^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we therefore have

$$B = [A]_U^U = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Then we can compute $B^n = \begin{pmatrix} 3^n & 3^{n-1}n \\ 0 & 3^n \end{pmatrix}$. Therefore $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3^{n-1}n \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 3^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3^n(n+1) \\ 3^{n-1}n \end{pmatrix}$ and $X_n = 3^{n-1}n$. □

4. Suppose that $T : V \to V$ is a linear transformation, that λ and μ are distinct real numbers, and that m and n are positive integers. Suppose that $\mathbf{v}$ is a vector in V such that $(T - \lambda I)^n(\mathbf{v}) = \mathbf{0}$ and $(T - \mu I)^m(\mathbf{v}) = \mathbf{0}$. Prove that $\mathbf{v} = \mathbf{0}$. You may wish to use the following steps:

   (a) Prove the assertion when $m = 1$.

   Solution. When $m = 1$ we have $T(\mathbf{v}) = \mu\mathbf{v}$. Therefore $(T - \lambda I)(\mathbf{v}) = \mu v - \lambda v = (\mu - \lambda)v$. But then $(T - \lambda I)^n(\mathbf{v}) = (\mu - \lambda)^n\mathbf{v}$. Therefore $(\mu - \lambda)^n\mathbf{v} = (T - \lambda I)^n\mathbf{v} = \mathbf{0}$. As $\mu - \lambda \neq 0$, this implies $\mathbf{v} = \mathbf{0}$. □

(b) Prove that $(T-\lambda I)\circ(T-\mu I)=(T-\mu I)\circ(T-\lambda I)$.

Solution. We have

$$(T-\lambda I)(T-\mu I)=T^2-(\lambda+\mu)T+\lambda\mu I=T^2-(\mu+\lambda)T+\mu\lambda I=(T-\mu I)(T-\lambda I).$$

$\square$

(c) If $m>1$, let $\mathbf{w}=(T-\mu I)^{m-1}(\mathbf{v})$. Show that $(T-\lambda I)^n(\mathbf{w})=\mathbf{0}$.

Solution. Using the previous part, $(T-\lambda I)^n(T-\mu I)^{m-1}=(T-\mu I)^{m-1}(T-\lambda I)^n$. Therefore:

$$\begin{aligned}(T-\lambda I)^n\mathbf{w}&=(T-\lambda I)^n(T-\mu I)^{m-1}\mathbf{v}\\&=(T-\mu I)^{m-1}(T-\lambda I)^n\mathbf{v}\\&=(T-\mu I)^{m-1}\mathbf{0}\\&=\mathbf{0}\end{aligned}$$

$\square$

(d) Use induction and the previous parts of the problem to prove that $\mathbf{v}=\mathbf{0}$.

Solution. The base case is $m=1$, which was covered in part (a). For the inductive step assume that, for all vectors $\mathbf{u}$, if $(T-\lambda I)^n\mathbf{u}=(T-\mu I)^{m-1}\mathbf{u}=\mathbf{0}$ then $\mathbf{u}=\mathbf{0}$. Let $\mathbf{w}=(T-\mu I)^{m-1}\mathbf{v}$. Then we have

$$\begin{aligned}(T-\mu I)\mathbf{w}&=(T-\mu I)^m\mathbf{v}=\mathbf{0}\\(T-\lambda I)\mathbf{w}&=\mathbf{0}\qquad\text{by part (c).}\end{aligned}$$

By part (a), this means that $\mathbf{w}=\mathbf{0}$. Therefore we have $(T-\lambda I)^n\mathbf{v}=(T-\mu I)^{m-1}\mathbf{v}=\mathbf{0}$, so by induction, $\mathbf{v}=\mathbf{0}$, as desired. $\square$