Problem Set #5

Math 2135 Spring 2020

Due Friday, April 17

1. There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the vectors

$$\mathbf{x}^1 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{x}^2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors with the following eigenvalues:

$$\lambda_1 = 7 \qquad \lambda_2 = -1 \qquad \lambda_3 = 5$$

Compute $T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (in the standard basis).

2. Let P_n be the vector space of polynomials of degree $\leq n$ and let $T : P_n \to P_n$ be the linear transformation $T(p)(x) = xp'(x)$. Find all eigenvalues and eigenvectors of T.

3. Define a sequence by the following recursive formula:

$$\begin{aligned} X_0 &= 0 \\ X_1 &= 1 \\ X_n &= 6X_{n-1} - 9X_{n-2} && \text{for } n \geq 2. \end{aligned}$$

Compute X_{2020}. Make sure to justify your answer with a proof! (Hint: the method we used for the Fibonacci numbers will not work exactly the same way here, but it will get you started. Look for a basis of generalized eigenvectors instead of a basis of eigenvectors.)

4. Suppose that $T : V \to V$ is a linear transformation, that λ and μ are distinct real numbers, and that m and n are positive integers. Suppose that $\mathbf{v}$ is a vector in V such that $(T - \lambda I)^n(\mathbf{v}) = \mathbf{0}$ and $(T - \mu I)^m(\mathbf{v}) = \mathbf{0}$. Prove that $\mathbf{v} = \mathbf{0}$. You may wish to use the following steps:

   (a) Prove the assertion when $m = 1$.

   (b) Prove that $(T - \lambda I) \circ (T - \mu I) = (T - \mu I) \circ (T - \lambda I)$.

   (c) If $m > 1$, let $\mathbf{w} = (T - \mu I)^{m-1}(\mathbf{v})$. Show that $(T - \lambda I)^n(\mathbf{w}) = \mathbf{0}$.

   (d) Use induction and the previous parts of the problem to prove that $\mathbf{v} = \mathbf{0}$.