Problem Set #4

Math 2135 Spring 2020

Due Friday, April 17

1. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & 2 \\ 1 & 1 & t \end{pmatrix}$$

(a) Find the inverse of A when $t = 1$.

Solution.

$$\begin{aligned}
&\operatorname{rref}\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 0 & -1 & -3 & 1 & 0 & -2 \\ 0 & 1 & 6 & 0 & 1 & 4 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 0 & -1 & -3 & 1 & 0 & -2 \\ 0 & 0 & 3 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 & 1 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 2 \\ 1 & 0 & 0 & 5/3 & 2/3 & 1/3 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 2/3 \\ 1 & 0 & 0 & 5/3 & 2/3 & 1/3 \end{array}\right) \\
=&\operatorname{rref}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/3 & 2/3 & 1/3 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 2/3 \end{array}\right)
\end{aligned}$$

$\square$

(b) Find all values of t such that A in invertible. Explain how you know that A is invertible for those values of t, and how you know it is not invertible for all other values.

Solution. We compute the determinant:

$$\det(A) = t(-6+4) - 1(4-4) + 1(2-3) = -2t - 1$$

We have an inverse if and only if $\det(A) \neq 0$, and this occurs if and only if t is not $-\frac{1}{2}$. $\square$

(c) For what values of t does the inverse of A have integer entries?

Solution. If C is the cofactor matrix of A then $A^{-1} = \det(A)^{-1}C^T$, so A has an inverse with integer entries if and only if $\det(A) = \pm 1$. This occurs if and only if $t = 0$ or $t = -1$. $\square$

2. In this question, we will study an unknown 4×5 matrix, A. We will write $\mathbf{u}^i$ for the columns of A and $\mathbf{v}_j$ for the rows of A:

$$A = \begin{pmatrix} \mathbf{u}^1 & \mathbf{u}^2 & \mathbf{u}^3 & \mathbf{u}^4 & \mathbf{u}^5 \end{pmatrix}$$

While we do not know what A is, we do know its reduced row echelon form:

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 3 & 0 & -5 \\ 0 & 1 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer the following questions using the matrix $\operatorname{rref}(A)$, above, and the column vectors $\mathbf{u}^1, \ldots, \mathbf{u}^5$. Justify your answers, briefly, and remember that $A \neq \operatorname{rref}(A)$.

(a) What is $\operatorname{rank}(A)$?

Solution. The rank is equal to the number of pivots in $\operatorname{rref}(A)$, which is 3. $\square$

(b) Find a basis for $\operatorname{null}(A)$, the null space of A.

Solution. The following vectors form a basis for $\operatorname{null}(A) = \operatorname{null}(\operatorname{rref}(A))$:

$$\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

$\square$

(c) Find a basis for $\operatorname{col}(A)$, the column space of A.

Solution. $\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^4$ form a basis for $\operatorname{col}(A)$ because the corresponding vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ form a basis for $\operatorname{col}(\operatorname{rref}(A))$. $\square$

(d) Compute the dimension of $\operatorname{null}(A^T)$, the left null space of A. Explain why it is not possible to find a basis for $\operatorname{null}(A^T)$ from the information given.

Solution. By the rank-nullity theorem, we have $\dim \mathrm{null}(A^T) + \mathrm{rank}(A^T) = 4$. We have already computed $\mathrm{rank}(A^T) = \mathrm{rank}(A) = 3$, so $\dim \mathrm{null}(A^T) = 1$. But we can't compute $\mathrm{null}(A^T)$ because the row operations that transform A to $\mathrm{rref}(A)$ correspond to column operations on A^T, and column operations can change the null space. □

(e) Find a basis for $\mathrm{col}(A^T)$, the row space of A.

Solution. Row operations do not change the row space, so the following vectors are a basis for the row space:

$$\begin{pmatrix} 1 & 0 & 3 & 0 & -5 \end{pmatrix}^T$$
$$\begin{pmatrix} 0 & 1 & -1 & 0 & 6 \end{pmatrix}^T$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 3 \end{pmatrix}^T$$

□

(f) Find a 3-element subset of $\{\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3, \mathbf{u}^4, \mathbf{u}^5\}$ that is linearly dependent.

Solution. The only soluiton is $\{\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3\}$. The corresponding columns of $\mathrm{rref}(A)$ are $\mathbf{e}^1$, $\mathbf{e}^2$, and $3\mathbf{e}^1 - \mathbf{e}^2$, which are linearly independent. Since the linear relationships between the columns of $\mathrm{rref}(A)$ are the same as the linear relationships between the corresponding columns of A, we have $\mathbf{u}^3 = 3\mathbf{u}^1 - \mathbf{u}^2$. □

3. Suppose that $T : V \to V$ is a linear transformation and that X and Y are bases of V. Prove that $\det([T]_X^X) = \det([T]_Y^Y)$. (Hint: write down the change of basis formula that relates $[T]_X^X$ and $[T]_Y^Y$.)

Solution. We have the change of basis formula:

$$[T]_Y^Y = [\mathrm{id}]_Y^X [T]_X^X [\mathrm{id}]_X^Y$$

Therefore

$$\det[T]_Y^Y = \det([\mathrm{id}]_Y^X) \det([T]_X^X) \det([\mathrm{id}]_X^Y).$$

But $[\mathrm{id}]_Y^X$ and $[\mathrm{id}]_X^Y$ are inverse to each other, so $\det([\mathrm{id}]_Y^X) \det([\mathrm{id}]_X^Y) = 1$. Substituting this into the above equation gives what we want:

$$\det[T]_X^X = \det[T]_Y^Y$$

□

4. Suppose that $S : U \to V$ and $T : V \to W$ are linear transformations. Prove that $\dim \ker(T \circ S) \leq \dim \ker(T) + \dim \ker(S)$. (Hint: restrict S to $\ker(T \circ S)$ to get a linear transformation $S' : \ker(T \circ S) \to \ker(T)$.)

Solution. Let S' be the restriction of S to $\ker(T \circ S)$. If $\mathbf{v} \in \ker(T \circ S)$ then $T(S'(\mathbf{v})) = T(S(\mathbf{v})) = \mathbf{0}$, so $\operatorname{image} S' \subset \ker(T)$ and in particular, $\dim \operatorname{image}(S') \leq \dim \ker(T)$. We also have $\ker(S') \subset \ker(S)$ because if $\mathbf{v} \in \ker(S')$ then $S(\mathbf{v}) = \mathbf{0}$, by definition. Thus $\dim \ker(S') \leq \dim \ker(S)$. (Actually $\ker(S) = \ker(S')$, but we won't need this.)

By the rank-nullity theorem, we now have

$$\dim \ker(T \circ S) = \dim \ker(S') + \dim \operatorname{image}(S') \leq \dim \ker(S) + \dim \ker(T).$$

$\square$