# Problem Set #4

Math 2135 Spring 2020

Due Friday, April 17

1. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & 2 \\ 1 & 1 & t \end{pmatrix}$$

   (a) Find the inverse of $A$ when $t = 1$.

   (b) Find all values of $t$ such that $A$ in invertible. Explain how you know that $A$ is invertible for those values of $t$, and how you know it is not invertible for all other values.

   (c) For what values of $t$ does the inverse of $A$ have integer entries?

2. In this question, we will study an unknown $4 \times 5$ matrix, $A$. We will write $\mathbf{u}^i$ for the columns of $A$ and $\mathbf{v}_j$ for the rows of $A$:

$$A = \begin{pmatrix} \mathbf{u}^1 & \mathbf{u}^2 & \mathbf{u}^3 & \mathbf{u}^4 & \mathbf{u}^5 \end{pmatrix}$$

   While we do not know what $A$ is, we do know its reduced row echelon form:

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 3 & 0 & -5 \\ 0 & 1 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   Answer the following questions using the matrix $\text{rref}(A)$, above, and the column vectors $\mathbf{u}^1, \dots, \mathbf{u}^5$. Justify your answers, briefly, and remember that $A \neq \text{rref}(A)$.

   (a) What is $\text{rank}(A)$?

   (b) Find a basis for $\text{null}(A)$, the null space of $A$.

   (c) Find a basis for $\text{col}(A)$, the column space of $A$.

   (d) Compute the dimension of $\text{null}(A^T)$, the left null space of $A$. Explain why it is not possible to find a basis for $\text{null}(A^T)$ from the information given.

   (e) Find a basis for $\text{col}(A^T)$, the row space of $A$.

   (f) Find a 3-element subset of $\{\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3, \mathbf{u}^4, \mathbf{u}^5\}$ that is linearly dependent.

3. Suppose that $T : V \to V$ is a linear transformation and that $X$ and $Y$ are bases of $V$. Prove that $\det([T]_X^X) = \det([T]_Y^Y)$. (Hint: write down the change of basis formula that relates $[T]_X^X$ and $[T]_Y^Y$.)

4. Suppose that $S : U \to V$ and $T : V \to W$ are linear transformations. Prove that $\dim \ker(T \circ S) \leq \dim \ker(T) + \dim \ker(S)$. (Hint: restrict $S$ to $\ker(T \circ S)$ to get a linear transformation $S' : \ker(T \circ S) \to \ker(T)$.)