# Problem Set \#3 

Math 2135 Spring 2020
Due Tuesday, March 10

1. (a) Put the following matrix in reduced row echelon form using a sequence of elementary row operations. In your answer, you should show all of your steps. You may combine consecutive steps as long the result of those steps does not depend on the order in which you perform them.

$$
A=\left(\begin{array}{ccccc}
0 & 3 & 4 & 0 & 2 \\
1 & 0 & 2 & 1 & 0 \\
-3 & -3 & -10 & 0 & 0 \\
2 & -6 & -4 & 1 & 2
\end{array}\right)
$$

(b) Express the solutions $\mathbf{x}$ to the equation $A \mathbf{x}=\mathbf{0}$ (where $A$ is as in the first part of this problem) as the span of a finite collection of vectors.
2. Determine for which values of $t$ the vector $\mathbf{v}=\left(\begin{array}{l}1 \\ 0 \\ 0 \\ t\end{array}\right)$ is in the span of the following vectors:

$$
\left(\begin{array}{c}
0 \\
4 \\
-2 \\
1
\end{array}\right) \quad\left(\begin{array}{l}
3 \\
5 \\
0 \\
2
\end{array}\right) \quad\left(\begin{array}{c}
5 \\
11 \\
-6 \\
4
\end{array}\right)
$$

Give a complete justification for your answer.
3. Recall that a subspace of a vector space $V$ is a subset $W \subset V$ with the following properties:

S1 $0 \in W$
S2 if $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u}+\mathbf{v} \in W$, and
S3 if $c \in \mathbb{R}$ and $\mathbf{v} \in W$ then $c \mathbf{v} \in W$.
Determine whether each of the following subsets $W$ of $V$ is a subspace. In the cases where $W$ is not a subspace, indicate which of the required properties of a subspace fails. In the pictures, $W$ is the region shown in red.
(a) $W=\{\mathbf{0}\}$
(d) $W=\varnothing$
(b)

(e) $W=V$
(f)

(c)

4. Let $V$ be a vector space and let $S: V \rightarrow \mathbb{R}$ and $T: V \rightarrow \mathbb{R}$ be linear transformations. Suppose that $\operatorname{ker}(S) \supset \operatorname{ker}(T)$. Prove that there is a real number $c$ such that $S=c T$.
(a) Prove the theorem in the case where $\operatorname{ker}(T)=V$.
(b) Now assume that $\operatorname{ker}(T) \neq V$. Explain why this assumption allows you to find a vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) \neq 0$.
(c) Prove that, if $\mathbf{w} \in V$, then there is a vector $\mathbf{u} \in \operatorname{ker}(T)$ and a scalar $a \in \mathbb{R}$ such that $\mathbf{w}=\mathbf{u}+a \mathbf{v}$ (where $\mathbf{v}$ is the vector from the last part). Make sure to explain why the choice of $\mathbf{v}$ in the last part was important.
(d) Finish the proof of the theorem. (Hint: start by calculating what $c$ must be.)

