Problem Set #2

Math 2135 Spring 2020

Due Saturday, February 22

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation given by the following formula:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_3 \\ x_1 - 4x_2 + x_3 \end{pmatrix}$$

   (a) Find the matrix $[T]_E^E$ in the standard bases.

   Solution.

$$\begin{pmatrix} 3 & 0 & -1 \\ 1 & -4 & 1 \end{pmatrix}$$

$\square$

   (b) Find all solutions to the equations $T(\mathbf{x}) = \mathbf{0}$.

   Solution. It is equivalent to find solutions to the following system of equations:

$$\begin{aligned} 3x_1 \qquad\quad - x_3 &= 0 \\ x_1 - 4x_2 + x_3 &= 0 \end{aligned}$$

   Adding the first equation to the second, we get

$$\begin{aligned} 3x_1 \qquad\quad - x_3 &= 0 \\ 4x_1 - 4x_2 \qquad\quad &= 0 \end{aligned}$$

   These give $x_1 = x_2$ and $x_3 = 3x_1$. Therefore the solutions are all vectors of the form

$$\begin{pmatrix} x_1 \\ x_1 \\ 3x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

   where x_1 is allowed to be any real number. $\square$

2. Let V be the vector space of straight-line motions in 3-dimensions. Suppose that W is a plane through the origin in V and that $X = \begin{pmatrix} \mathbf{x}^1 & \mathbf{x}^2 & \mathbf{x}^3 \end{pmatrix}$ is a basis of V such that $\mathbf{x}^1$ and $\mathbf{x}^2$ are in W and $\mathbf{x}^3$ is perpendicular to W. Let $T : V \to V$ be the reflection across the plane W. Find the matrix $[T]_X^X$ of T in the basis X.

Solution. By definition, $T(\mathbf{x}^1) = \mathbf{x}^1$ and $T(\mathbf{x}^2) = \mathbf{x}^2$ and $T(\mathbf{x}^3) = -\mathbf{x}^3$. Therefore

$$[T]_X^X = \begin{pmatrix} [T(\mathbf{x}^1)]_X & [T(\mathbf{x}^2)]_X & [T(\mathbf{x}^3)]_X \end{pmatrix} = \begin{pmatrix} [\mathbf{x}^1]_X & [\mathbf{x}^2]_X & [\mathbf{x}^3]_X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

□

3. Let V be the vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) = a\exp(x) + b\cos(x) + c\sin(x)$$

for some real numbers a, b, and c. (The function exp refers to the exponential, $\exp(x) = e^x$.) Let F be the basis $\begin{pmatrix} \exp & \cos & \sin \end{pmatrix}$ of V. Let $T : V \to V$ be the linear transformation $T(f) = f + f' + 2f''$ (where f' is the derivative of f). You may use the linearity of the derivative in your proof.

(i) Prove that T is a linear transformation.

Solution. For real numbers a and b, we have

$$\begin{aligned} T(af + bg) &= (af + bg) + (af + bg)' + 2(af + bg)'' \\ &= af + bg + af' + bg' + 2af'' + 2bg'' \\ &= a(f + f' + 2f'') + b(g + g' + 2g'') \\ &= aT(f) + bT(g) \end{aligned}$$

This shows that T preserves scalar multiplication and vector addition, so T is linear. □

(ii) Compute $[T]_F^F$.

Solution. We have $T(\exp) = \exp + \exp + 2\exp = 4\exp$ since $\exp' = \exp$. We have $T(\cos) = \cos - \sin - 2\cos = -\cos - \sin$ since $\cos' = -\sin$ and $\cos'' = -\cos$. We have $T(\sin) = \sin + \cos - 2\sin = \cos - \sin$. Thus

$$\begin{aligned} [T]_X^X &= \begin{pmatrix} [T(\exp)]_X & [T(\cos)]_X & [T(\sin)]_X \end{pmatrix} \\ &= \begin{pmatrix} [4\exp]_X & [-\cos - \sin]_X & [\cos - \sin]_X \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \end{aligned}$$

□

4. For each $i = 1, \ldots, n$, let $\mathbf{e}_i : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation given by the following formula:

$$\mathbf{e}_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i$$

(a) Find the matrix of each $\mathbf{e}_i$.

Solution.

$$[\mathbf{e}_i] = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

with the 1 in the i-th column. $\square$

(b) Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ be the vector space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$. Prove that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent in $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$. (You do not need to verify that $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is a vector space.)

Solution. Suppose $\sum a^i \mathbf{e}_i = \mathbf{0}$. Then for every vector $\mathbf{v} \in \mathbf{R}^n$, we have $\sum a^i \mathbf{e}_i(\mathbf{v}) = 0$. In particular, $\sum a^i \mathbf{e}_i(\mathbf{e}^j) = 0$ for each standard basis vector $\mathbf{e}^j$. Since $\mathbf{e}_i(\mathbf{e}^j) = 0$ for $i \neq j$ and $\mathbf{e}_i(\mathbf{e}^j) = 1$ for $i = j$ this equation simplifies to $a^j = 0$. This holds for all j, so we conlcude the only solution to $\sum a^j \mathbf{e}_j = \mathbf{0}$ is $a^1 = \cdots = a^n = 0$. Therefore the $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent. $\square$

(c) Prove that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

Solution. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is a linear transformation. We will prove that $f = \sum_j f(\mathbf{e}^j)\mathbf{e}_j$. We have to check that f and $\sum_j f(\mathbf{e}^j)\mathbf{e}_j$ take the same values on every vector $\mathbf{v}$ in $\mathbb{R}^n$. Suppose that $\mathbf{v} = a_1\mathbf{e}^1 + \cdots + a_n\mathbf{e}^n$. Then we have

$$\begin{aligned}
f(\mathbf{v}) &= f(a_1\mathbf{e}^1 + \cdots + a_n\mathbf{e}^n) \\
&= a_1 f(\mathbf{e}^1) + \cdots + a_n f(\mathbf{e}^n) \\
\sum f(\mathbf{e}^j)\mathbf{e}_j(\mathbf{v}) &= \sum_j f(\mathbf{e}^j)\mathbf{e}_j(\sum_i a_i\mathbf{e}^i) \\
&= \sum_{i,j} f(\mathbf{e}^j) a_i \mathbf{e}_j(\mathbf{e}^i) \\
&= f(\mathbf{e}^1)a_1 + \cdots + f(\mathbf{e}^n)a_n
\end{aligned}$$

Therefore $f = \sum_j f(\mathbf{e}^j)\mathbf{e}_j$. $\square$

(d) Conclude that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis for $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

Solution. We have shown that they span $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ and are linearly independent. $\square$