Problem Set #2

Math 2135 Spring 2020

Due Saturday, February 22

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation given by the following formula:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_3 \\ x_1 - 4x_2 + x_3 \end{pmatrix}$$

   (a) Find the matrix $[T]_E^E$ in the standard bases.

   (b) Find all solutions to the equations $T(\mathbf{x}) = \mathbf{0}$.

2. Let V be the vector space of straight-line motions in 3-dimensions. Suppose that W is a plane through the origin in V and that $X = \begin{pmatrix} \mathbf{x}^1 & \mathbf{x}^2 & \mathbf{x}^3 \end{pmatrix}$ is a basis of V such that $\mathbf{x}^1$ and $\mathbf{x}^2$ are in W and $\mathbf{x}^3$ is perpendicular to W. Let $T : V \to V$ be the reflection across the plane W. Find the matrix $[T]_X^X$ of T in the basis X.

3. Let V be the vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) = a\exp(x) + b\cos(x) + c\sin(x)$$

   for some real numbers a, b, and c. (The function exp refers to the exponential, $\exp(x) = e^x$.) Let F be the basis $\begin{pmatrix} \exp & \cos & \sin \end{pmatrix}$ of V. Let $T : V \to V$ be the linear transformation $T(f) = f + f' + 2f''$ (where f' is the derivative of f). You may use the linearity of the derivative in your proof.

   (i) Prove that T is a linear transformation.

   (ii) Compute $[T]_F^F$.

4. For each $i = 1, \ldots, n$, let $\mathbf{e}_i : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation given by the following formula:

$$\mathbf{e}_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i$$

   (a) Find the matrix of each $\mathbf{e}_i$.

   (b) Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ be the vector space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$. Prove that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent in $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$. (You do not need to verify that $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is a vector space.)

   (c) Prove that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

   (d) Conclude that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis for $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$.