Graded Problem Set #1

Math 2135 Spring 2020

Due Friday, January 24

1. Express each of the following vectors as a single column vector whose entries are real numbers:

   (i)
   $$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$$

   Solution.
   $$\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

   □

   (ii)
   $$\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   Solution.
   $$\begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

   □

2. Solve the following equation for a vector $\mathbf{x}$ in $\mathbb{R}^2$:
   $$\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -4 \\ 13 \end{pmatrix}$$

   Solution. Write $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then we get the system of linear equations:
   $$\begin{aligned} 3x_1 + x_2 &= -4 \\ -x_1 + 2x_2 &= 13 \end{aligned}$$

Add 3 times the second equation equation to the first. We get:

$$\begin{aligned} 7x_2 &= 35 \\ -x_1 + 2x_2 &= 13 \end{aligned}$$

The first equation gives $x_2 = 5$. Then substitute into the second equation to get

$$-x_1 + 10 = 13$$

and solve to get $x_1 = -3$. Thus $\mathbf{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. $\square$

3. Let V be the set of all 2×2 matrices with real entries, with zero vector, vector addition, and scalar multiplication defined as follows:

V0) $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

V1) $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_3 + y_3 & x_4 + y_4 \end{pmatrix}$

V2) $c \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} cx_1 & cx_2 \\ cx_3 & cx_4 \end{pmatrix}$

Prove the distributive property for scalar multiplication over vector addition ($c(\mathbf{x}+\mathbf{y}) = (c\mathbf{x}) + (c\mathbf{y})$ for all $c \in \mathbb{R}$, $\mathbf{x} \in V$, and $\mathbf{y} \in V$). You may use any true properties you know about the real numbers, but no properties of 2×2 matrices other than the ones given above.

Solution. Suppose that $c \in \mathbb{R}$ and $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$. Then we have the following:

$$\begin{aligned}
c(\mathbf{x} + \mathbf{y}) &= c\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}\right) \\
&= c\begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_3 + y_3 & x_4 + y_4 \end{pmatrix} && \text{definition of vector addition} \\
&= \begin{pmatrix} c(x_1 + y_1) & c(x_2 + y_2) \\ c(x_3 + y_3) & c(x_4 + y_4) \end{pmatrix} && \text{definition of scalar multiplication} \\
&= \begin{pmatrix} cx_1 + cy_1 & cx_2 + cy_2 \\ cx_3 + cy_3 & cx_4 + cy_4 \end{pmatrix} && \text{distributive property in } \mathbb{R} \\
&= \begin{pmatrix} cx_1 & cx_2 \\ cx_3 & cx_4 \end{pmatrix} + \begin{pmatrix} cy_1 & cy_2 \\ cy_3 & cy_4 \end{pmatrix} && \text{definition of vector addition} \\
&= \left(c\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) + \left(c\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}\right) && \text{definition of scalar multiplication} \\
&= (c\mathbf{x}) + (c\mathbf{y})
\end{aligned}$$

$\square$

4. Prove that if V is a (real) vector space with finitely many elements then V has exactly one element.

Solution. Suppose that $\mathbf{v}$ is a vector in V and that V is a finite set. Since there are infinitely many real numbers and only finitely many elements of V, there must be at least two different real numbers such a and b such that $a\mathbf{v} = b\mathbf{v}$. But then $(a-b)\mathbf{v} = \mathbf{0}$. Since $a \neq b$, the number $a - b$ is invertible, so $\mathbf{v} = (a-b)^{-1}(a-b)\mathbf{v} = (a-b)^{-1}\mathbf{0} = \mathbf{0}$. Therefore $\mathbf{v} = \mathbf{0}$. This shoes that every vector in V is equal to $\mathbf{0}$, so V has only one element. $\square$