Handout #7

Math 2135 Spring 2020

Friday, April 3

In class we proved the following formulas for the determinant:

(a) The determinant is linear in each column:

$$\det\begin{pmatrix}\mathbf{u}^1 & \cdots & (a\mathbf{v}+b\mathbf{w}) & \cdots & \mathbf{u}^n\end{pmatrix} = a\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{v} & \cdots & \mathbf{u}^n\end{pmatrix} + b\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{w} & \cdots & \mathbf{u}^n\end{pmatrix}$$

(b) If *adjacent* columns of A are repeated then $\det(A) = 0$:

$$\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{v} & \mathbf{v} & \cdots & \mathbf{u}^n\end{pmatrix} = 0$$

(c) The determinant of the identity matrix is 1:

$$\det\begin{pmatrix}a_1\mathbf{e}^1 & \cdots & a_n\mathbf{e}^n\end{pmatrix} = a_1\cdots a_n$$

1. Use these properties to demonstrate the following additional properties of the determinant:

   (a) Swapping *adjacent* columns of A changes the sign of $\det(A)$:

   $$\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{v} & \mathbf{w} & \cdots & \mathbf{u}^n\end{pmatrix} = -\det\begin{pmatrix}u^1 & \cdots & \mathbf{w} & \mathbf{v} & \cdots & \mathbf{u}^n\end{pmatrix}$$

   (b) Swapping *any* two columns of A changes the sign of $\det(A)$ (hint: use the last part an odd number of times):

   $$\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{v} & \cdots & \mathbf{w} & \cdots & \mathbf{u}^n\end{pmatrix} = -\det\begin{pmatrix}u^1 & \cdots & \mathbf{w} & \cdots & \mathbf{v} & \cdots & \mathbf{u}^n\end{pmatrix}$$

   (c) If any column of A is repeated then $\det(A) = 0$:

   $$\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{v} & \cdots & \mathbf{v} & \cdots & \mathbf{u}^n\end{pmatrix} = 0$$

   (d) Adding a multiple of one column of A to another does not change the determinant of A:

   $$\det\begin{pmatrix}\mathbf{u}^1 & \cdots & \mathbf{u}^i & \cdots & \mathbf{u}^j + c\mathbf{u}^i & \cdots & \mathbf{u}^n\end{pmatrix} = \det\begin{pmatrix}u^1 & \cdots & \mathbf{u}^i & \cdots & \mathbf{u}^j & \cdots & \mathbf{u}^n\end{pmatrix}$$

For the next few exercises, you will need to use the fact that

$$\det(AB) = \det(A)\det(B)$$

We will prove this fact using your work above in class on Wednesday.

2. (LADW, Chapter 3, §3, #3.5) Suppose that A is a square matrix such that $A^n = 0$ for some integer $n \geq 1$. Prove that $\det(A) = 0$.

3. Suppose that A is a matrix of real numbers such that A^n is the identity matrix. What are the possibilities for $\det(A)$?

4. Suppose $T : V \to V$ is a linear transformation and X and Y are bases of V. Prove that $\det([T]_X^X) = \det([T]_Y^Y)$.

5. This exercise gives a geometric interpretation of the sign of the determinant. Let $\mathrm{GL}_n(\mathbb{R})$ be the set of invertible $n \times n$ matrices whose entries are real numbers. This is a subset of the set $\mathbb{R}^{n\times n}\mathbb{R}^{n^2}$

   (a) Show that $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ is a continuous function (hint: sums and products of continuous functions are continuous). Conclude that there can be no continuous path in $\mathrm{GL}_n(\mathbb{R})$ connecting a matrix with positive determinant to a matrix with negative determinant.

   (b) Show that there is a path in $\mathrm{GL}_n(\mathbb{R})$ connecting

   $$\begin{pmatrix} \mathbf{u}^1 & \cdots & \mathbf{u}^i & \cdots & \mathbf{u}^j & \cdots & \mathbf{u}^n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{u}^1 & \cdots & \mathbf{u}^i & \cdots & \mathbf{u}^j + c\mathbf{u}^i & \cdots & \mathbf{u}^n \end{pmatrix}$$

   and that there is a path in $\mathrm{GL}_n(\mathbf{R})$ connecting

   $$\begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_j \\ \vdots \\ \mathbf{u}_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_j + c\mathbf{u}_i \\ \vdots \\ \mathbf{u}_n \end{pmatrix}$$

   (c) Show that every matrix A in $\mathrm{GL}_n(\mathbb{R})$ can be connected by a path to either

   $$I = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

   (d) Conclude that $A \in \mathrm{GL}_n(\mathbb{R})$ can be connected by a path to the identity matrix if and only if $\det(A) > 0$. Otherwise $\det(A) < 0$.