Handout #4

Math 2135 Spring 2020

Friday, March 13

1. For which values of c is the following matrix invertible:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & c \end{pmatrix}$$

Solution. Compute the RREF.

$$\operatorname{rref}\begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & c \end{pmatrix} = \operatorname{rref}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & c-2 \end{pmatrix} = \operatorname{rref}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & c-2 \end{pmatrix}$$

This has a pivot in every row and column if and only if $c \neq 2$. □

2. Suppose that A is a 4×4 matrix and let I be the 4×4 identity matrix. Assume that

$$\operatorname{rref}(A|I) = \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 9 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 11 & 12 \end{array}\right)$$

(a) What is $\operatorname{rref}(A)$?

Solution.

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

□

(b) Does A have a left inverse?

Solution. No: it does not have a pivot in every column. □

(c) Does A have a right inverse?

Solution. No: it does not have a pivot in every row. □

(d) Find a basis for $\operatorname{null}(A)$.

Solution.

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

□

(e) Find a matrix Z such that $ZA = \mathrm{rref}(A)$.

(f) Use the reduced row echelon form above to find a matrix B such that $\mathrm{col}(A) = \mathrm{null}(B)$.

Solution.

$$B = \begin{pmatrix} 1 & 0 & 9 & 10 \\ 0 & 1 & 11 & 12 \end{pmatrix}$$

Write $\mathrm{rref}(A|I) = (A'|Z)$. Then $A\mathbf{x} = \mathbf{y}$ if and only if $A'\mathbf{x} = Z\mathbf{y}$. If $B\mathbf{y} = \mathbf{0}$ then this system is equivalent to

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 0 & 4 & 5 \\ 0 & 0 & 7 & 8 \end{pmatrix} \mathbf{y}$$

This definitely has a solution because the matrix on the left is in RREF with a pivot in every row. □

(g) Find a basis for $\mathrm{col}(A)$.

Solution.

$$\begin{pmatrix} -9 \\ -11 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -10 \\ -12 \\ 0 \\ 1 \end{pmatrix}$$

□

3. Do Exercise 5.6 of LADW, Chapter 2, §5: Show that the system of vectors

$$\mathbf{v}^1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 5 \\ -3 \end{pmatrix}, \qquad \mathbf{v}^2 = \begin{pmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}^3 = \begin{pmatrix} 1 \\ 1 \\ 50 \\ -921 \\ 0 \end{pmatrix}$$

is linearly independent by completing it to a basis of $\mathbb{R}^5$.

Solution. Add the column vectors $\mathbf{e}^1$ and $\mathbf{e}^3$. Then by column operations, can replace $\mathbf{v}^2$ with $\frac{1}{2}(\mathbf{v}^2 - 3\mathbf{e}^1) = \mathbf{e}^2$. Then by column operations, we can replace $\mathbf{v}^3$ with $\mathbf{e}^4 = -\frac{1}{921}(\mathbf{v}^3 - \mathbf{e}^1 - \mathbf{e}^2 - 50\mathbf{e}^3)$. Once we have $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4$, we can get $\mathbf{e}^5$ by subtracting multiples of these from $\mathbf{v}^1$. Therefore we have an linearly independent collection, and any subcollection of a linearly independent collection of vectors is independent. □