Handout #2

Math 2135 Spring 2020

Friday, February 21

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a polynomial of degree ≤ 2. Write a formula for $f'(0)$, $f'(1)$, and $f'(2)$ in terms of $f(0)$, $f(1)$, and $f(2)$. The following steps outline a solution to this problem:

   (a) Find polynomials f^0, f^1, and f^2 of degree ≤ 2 such that $f^i(j) = 1$ if $i = j$ and $f^i(j) = 0$ if $i \ne j$. That is we want polynomials f^0, f^1, and f^2 such that

   $$\begin{array}{lll} f^0(0) = 1 & f^1(0) = 0 & f^2(0) = 0 \\ f^0(1) = 0 & f^1(1) = 1 & f^2(1) = 0 \\ f^0(2) = 0 & f^1(2) = 0 & f^2(2) = 1 \end{array}$$

   Hint: try $(x-1)(x-2)$.

   (b) Prove that $F = \begin{pmatrix} f^0 & f^1 & f^2 \end{pmatrix}$ is a basis of the vector space P_2 of polynomials of degree ≤ 2. Hint: to solve $g = af^0 + bf^1 + cf^2$, try plugging in $x = 0$ to both sides.

   (c) Let $X = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}$ be the usual basis of P_2. Calculate the change of basis matrices $[\mathrm{id}]_X^F$ and $[\mathrm{id}]_F^X$. Hint: the hint for the previous part may help.

   (d) Let $D : P_2 \to P_2$ be the linear transformation that sends $f \in P_2$ to its derivative, f' (with respect to x). Calculate $[F]_X^X$.

   (e) Complete the problem.

   (f) How would you generalize this problem to polynomials of degree $\le n$?

2. (LADW, Chapter 1, Exercise 6.11) Find the matrix (in the standard basis, E) of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates counterclockwise by an angle θ around the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

   (a) Find a basis $X = \begin{pmatrix} \mathbf{x}^1 & \mathbf{x}^2 & \mathbf{x}^3 \end{pmatrix}$ of $\mathbb{R}^3$ such that $[T]_X^X$ is easy to calculate. (Hints: two vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ are perpendicular if $ad + be + cf = 0$; the cross product $\begin{pmatrix} bf - ce \\ -af + cd \\ ae - bd \end{pmatrix}$ is always perpendicular to both.)

(b) Calculate $[T]_X^X$.

(c) Calculate whichever of the change of basis matrices $[\mathrm{id}]_E^X$ and $[\mathrm{id}]_X^E$ is easier. You don't have to calculate both.

(d) Complete the problem in terms of $[\mathrm{id}]_E^X$ and $[\mathrm{id}]_X^E$ and $[T]_X^X$.