Exam #1

Math 2135 Spring 2020

Friday, February 7

Work independently. Write your answers on separate paper. Submit only your solutions, not this problem page.

1. The following questions refer to the diagram on the second page.

   (a) Find two linearly dependent vectors in the diagram, no one of which is linearly dependent by itself.

   Solution. $(\mathbf{v}^2, \mathbf{v}^4)$ or $(\mathbf{v}^3, \mathbf{v}^5)$ $\square$

   (b) Find all bases of V that consist only of vectors depicted in the figure. Order does not matter in this answer, so you only have to include one of $\begin{pmatrix}\mathbf{v}^1 & \mathbf{v}^2\end{pmatrix}$ and $\begin{pmatrix}\mathbf{v}^2 & \mathbf{v}^1\end{pmatrix}$.

   Solution. $(\mathbf{v}^1, \mathbf{v}^2), (\mathbf{v}^1, \mathbf{v}^3), (\mathbf{v}^1, \mathbf{v}^4), (\mathbf{v}^1, \mathbf{v}^5), (\mathbf{v}^2, \mathbf{v}^3), (\mathbf{v}^2, \mathbf{v}^5), (\mathbf{v}^3, \mathbf{v}^4), (\mathbf{v}^4, \mathbf{v}^5)$ $\square$

   (c) Let X be the basis $\begin{pmatrix}\mathbf{v}^1 & \mathbf{v}^2\end{pmatrix}$ of V. Find the coordinates $[\mathbf{v}^3]_X$ of $\mathbf{v}^3$ in the basis X.

   Solution. We have $\mathbf{v}^3 = -\frac{5}{3}\mathbf{v}^1 + 2\mathbf{v}^4$ so $[\mathbf{v}^3]_X = \begin{pmatrix} -5/3 \\ 2 \end{pmatrix}$. $\square$

2. Let V be a vector space containing linearly independent vectors $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^n$. Determine whether the vectors

$$\begin{aligned} \mathbf{x}^1 &= \mathbf{v}^1 - \mathbf{v}^2 \\ \mathbf{x}^2 &= \mathbf{v}^2 - \mathbf{v}^3 \\ \mathbf{x}^3 &= \mathbf{v}^3 - \mathbf{v}^4 \\ &\vdots \qquad \vdots \\ \mathbf{x}^{n-1} &= \mathbf{v}^{n-1} - \mathbf{v}^n \\ \mathbf{x}^n &= \mathbf{v}^n - \mathbf{v}^1 \end{aligned}$$

   are linearly dependent or linearly independent and justify your answer.

   Solution. We have

$$\mathbf{x}^1 + \mathbf{x}^2 + \cdots + \mathbf{x}^n = \mathbf{0}$$

   so these vectors are linearly dependent. $\square$

3. Consider the following three vectors in $\mathbb{R}^3$:

$$\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ t \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

For what values of t are these vectors linearly independent? For what values are they linearly dependent? Justify your answer.

Solution. These vectors are linearly dependent if and only if we can express one vector as a linear combination of the others. Neither of $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ nor $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ is a multiple of the other, so if one vector is to be a linear combination of the others, we must have

$$\begin{pmatrix} 3 \\ t \\ 2 \end{pmatrix} = a \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

for some real numbers a and b. We therefore must have $3 = b$ and $2 = -a - 2b$, so $a = -8$. Plugging these in to the equation $t = 2a$, we get $t = 2(-8) = -16$. If t is anything other than -16 then there will be no a and b that solve the vector equation above and then the vectors will be linearly independent. $\square$

4. Suppose that $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linearly independent vectors in a vector space V and $\mathbf{w}$ is a vector in V that is not a linear combination of $\mathbf{x}^1, \ldots, \mathbf{x}^n$. Show that the vectors $\mathbf{x}^1 + \mathbf{w}, \ldots, \mathbf{x}^n + \mathbf{w}$ are linearly independent.

Solution. Suppose $\sum a_i(\mathbf{x}^i + \mathbf{w}) = \mathbf{0}$. We want to show that the only solution to this equation is $a_1 = \cdots = a_n = 0$. If $a_1, \ldots, a_n$ are a solution to this equation then $\sum a_i \mathbf{x}^i = -\sum a_i \mathbf{w}$. If $\sum a_i \neq 0$ then this says that $\mathbf{w}$ is a linear combination of $\mathbf{x}^1, \ldots, \mathbf{x}^n$, and we have assumed that it is not. Therefore $\sum a_i = 0$ and we have

$$\sum a_i \mathbf{x}^i = \mathbf{0}$$

But $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linearly independent so $a_1 = \cdots = a_n = 0$, as we required. We may therefore conclude that $\mathbf{x}^1 + \mathbf{w}, \ldots, \mathbf{x}^n + \mathbf{w}$ are linearly independent. $\square$

Second solution. Since $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linearly independent and $\mathbf{w}$ is not a linear combination of $\mathbf{x}^1, \ldots, \mathbf{x}^n$, the longer list $\mathbf{x}^1, \ldots, \mathbf{x}^n, \mathbf{w}$ is linearly independent.

To prove that the list $\mathbf{x}^1 + \mathbf{w}, \ldots, \mathbf{w}^n + \mathbf{w}$ is linearly independent, we must show that the only scalars $a_1, \ldots, a_n$ such that

$$a_1(\mathbf{x}^1 + \mathbf{w}) + \cdots + a_n(\mathbf{x}^n + \mathbf{w}) = \mathbf{0}$$

are $a_1 = \cdots = a_n = 0$. This equation rearranges to

$$a_1\mathbf{x}^1 + \cdots + a_n\mathbf{x}^n + (a_1 + \cdots + a_n)\mathbf{w} = \mathbf{0}$$

Since $\mathbf{x}^1, \ldots, \mathbf{x}^n, \mathbf{w}$ are linearly independent, we get

$$a_1 = \cdots = a_n = a_1 + \cdots + a_n = 0$$

and in particular,

$$a_1 = \cdots = a_n = 0,$$

as required. $\square$

Let V be the vector space of straight-line motions in the plane. The diagram below shows several vectors in V: