Problem 1. Determine whether the following collection of vectors in $\mathbb{R}^4$ is linearly independent or dependent.

$$\begin{pmatrix} 3 \\ 1 \\ 2 \\ -4 \end{pmatrix} \quad \begin{pmatrix} 9 \\ 0 \\ 6 \\ -3 \end{pmatrix} \quad \begin{pmatrix} -3 \\ -2 \\ -4 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 7 \\ 6 \\ 0 \\ 3 \end{pmatrix}$$

Warning: a small arithmetic error could lead to an incorrect answer! Try to solve this problem in a way that requires as little arithmetic as possible

Problem 2. Suppose that $\vec{u}^1$, $\vec{u}^2$, $\vec{u}^3$, $\vec{u}^4$, $\vec{u}^5$ are vectors in $\mathbb{R}^4$ such that the matrix $M = \begin{pmatrix} \vec{u}^1 & \vec{u}^2 & \vec{u}^3 & \vec{u}^4 & \vec{u}^5 \end{pmatrix}$ is row equivalent to the following matrix:

$$\begin{pmatrix} 1 & 0 & 3 & 0 & -5 \\ 0 & 1 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Which of the following statements are necessarily true? Explain your answers briefly.

(a) Are the vectors $\vec{u}^1, \vec{u}^2, \vec{u}^3, \vec{u}^4$ are linearly independent? Justify your answer, briefly.

(b) Is it true that $\vec{u}^5 = -5\vec{u}^1 + 6\vec{u}^2 + 3\vec{u}^4$? Justify your answer, briefly.

(c) Find *all* maximal linearly independent subsets of $\{\vec{u}^1, \vec{u}^2, \vec{u}^3, \vec{u}^4, \vec{u}^5\}$. No justification is required.

Problem 3. Suppose that F is a field and that $\vec{u}^1, \ldots, \vec{u}^n$ are vectors in F^m. Let

$$M = \begin{pmatrix} \vec{u}^1 & \vec{u}^2 & \vec{u}^3 & \cdots & \vec{u}^n \end{pmatrix}$$

be the matrix with n rows and m columns whose i-th column is $\vec{u}^i$. For each of the properties on the right in the table below, there is one matrix in the left column that must be equivalent to M by row and column operations. Indicate which matrices correspond to which properties by drawing a line connecting a matrix to the corresponding property.

M is equivalent by row and column operations to...	if and only if $\vec{u}^1, \ldots, \vec{u}^m$ are...
$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$	linearly dependent and span F^m.
$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$	a basis of F^m.
$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$	linearly independent and do not span F^m.
$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$	linearly dependent and do not span F^m.

Problem 4. Let V be a vector space over a field F. Suppose that S is a set of vectors in V and that $f : V \to F$ is a linear functional such that $f(\vec{v}) = 0$ for all $\vec{v} \in S$. Let $\vec{w}$ be a vector in V such that $f(\vec{w}) \neq 0$. Prove that $\vec{w}$ is not in the span of S.

Solution. We prove the contrapositive: if $\vec{w} \in \operatorname{span} S$ then $f(\vec{w}) = 0$. Suppose that $\vec{w} \in \operatorname{span} S$. Then $\vec{w} = \sum_{i=1}^{n} a_i \vec{v}^i$ for some $\vec{v}^1, \ldots, \vec{v}^n \in S$ and some $a_1, \ldots, a_n \in F$. Therefore

$$\begin{aligned} f(\vec{w}) &= f(\sum_{i=1}^{n} a_i \vec{v}^i) && \text{substitution for } \vec{w} \\ &= \sum_{i=1}^{n} a_i f(\vec{v}^i) && f \text{ is linear} \\ &= \sum_{i=1}^{n} a_i 0 && \vec{v}^i \in S \text{ so } f(\vec{v}^i) = 0 \text{ for all } i \\ &= 0 \end{aligned}$$

as requried. $\square$