Problem 1. (i) Compute the determinant of the following matrix:

$$\begin{pmatrix} 6 & 15 & 0 \\ 1 & 3 & x \\ 0 & x+1 & 2 \end{pmatrix}$$

(ii) For which values of x is the above matrix invertible?

Problem 2. Let M be an $n \times n$ matrix of real numbers such that $M^p = I_n$ for some positive integer n. What are the possible values of $\det(M)$? Justify your answer.

Problem 3. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $[\varphi]_E^E = \begin{pmatrix} 42 & -30 \\ 60 & -43 \end{pmatrix}$ where $E = \{\vec{e}^1, \vec{e}^2\}$ is the standard basis.

(i) Find a basis S of $\mathbb{R}^2$ such that $[\varphi]_S^S$ is diagonal.

(ii) Compute $([\varphi]_E^E)^{2019}$. (The entries in the answer have several hundred digits, but you should be able to express each entry succinctly as $ax^\alpha + by^\beta$ for some integers a, b, x, y, α, and β.)

Problem 4. Let $\varphi : V \to V$ be a linear transformation and let $\vec{v}$ and $\vec{w}$ be eigenvectors of φ with eigenvalues λ and μ, respectively. Prove that $\vec{v} + \vec{w}$ is an eigenvector of φ if and only if $\lambda = \mu$.

Problem 5. (i) Let M be an $n \times n$ matrix. The transpose M^t of M is the matrix obtained by reflecting M over its diagonal. If the entry in the i-th row and j-th column of M is a_{ij} then the entry in the i-th row and j-th column of M^t is a_{ji}. Prove that $\det M^t = \det M$.

(ii) M is called *skew-symmetric* if $M^t = -M$. Show that every $n \times n$ skew-symmetric matrix M has $\det M = 0$ if n is odd.