Problem 1. For certain bases S and T of $\mathbb{R}^3$, one change of basis matrix is given by the following formula:

$$[\mathrm{id}]_T^S = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

Find the other chnage of basis matrix, $[\mathrm{id}]_S^T$.

Problem 2. Let V be the vector space over $\mathbb{R}$ of all functions $\vec{f} : \mathbb{R} \to \mathbb{R}$. Let

$$S = \{\cos(x), \sin(x), \cos(2x), \sin(2x)\}$$

and let W be the span of S. Let $D : W \to W$ be the linear transformation $D(f) = \frac{df}{dx}$. Find the matrix $[D]_S^S$ of D in the basis S (you do not need to prove that D is a basis).

Problem 3. For each of the following linear transformations $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$, find a basis for $\ker \varphi$ and $\operatorname{im} \varphi$.

(i) $[\varphi]_E^E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(ii) $[\varphi]_E^E = \begin{pmatrix} 3 & -1 & 2 & 0 \\ -6 & 2 & -4 & 0 \\ 0 & 2 & 1 & -1 \\ 3 & 1 & 3 & -1 \end{pmatrix}$

Problem 4. Let $\vec{u}^1, \vec{u}^2, \vec{u}^3$ be the following vectors in $\mathbb{R}^3$:

$$\vec{u}^1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \qquad \vec{u}^2 = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \qquad \vec{u}^3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

There is a linear transformation $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties:

$$\varphi(\vec{u}^1) = \vec{u}^2 \qquad \varphi(\vec{u}^2) = \vec{u}^3 \qquad \varphi(\vec{u}^3) = \vec{u}^1$$

(i) Find the matrix $[\varphi]_S^S$ of φ in the basis $S = \{\vec{u}^1, \vec{u}^2, \vec{u}^3\}$ of $\mathbb{R}^3$.

(ii) Find the matrix $[\varphi]_E^E$ of φ in the standard basis $E = \{\vec{e}^1, \vec{e}^2, \vec{e}^3\}$ of $\mathbb{R}^3$.

(iii) Suppose that T is another, unknown, basis of $\mathbb{R}^3$ and let $M = [\varphi]_T^T$. Compute M^3 (that is, $M \cdot M \cdot M$) and justify your answer.

Problem 5. Let U, V, and W be vector spaces over a field F and let $\varphi : V \to W$ and $\psi : U \to V$ be linear transformations.

(i) Suppose that ψ is surjective. Prove that $\operatorname{im}(\varphi\psi) = \operatorname{im}(\varphi)$.

(ii) Suppose that ψ is bijective. Prove $\dim \ker(\varphi\psi) = \dim \ker(\varphi)$.