Problem 1. Compute the dimension of the subspace V of $\mathbb{R}^5$ spanned by the following vectors:

$$\begin{pmatrix} -2 \\ -4 \\ -4 \\ -3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 7 \\ 2 \\ -1 \\ 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 0 \\ -2 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 6 \\ 4 \\ 2 \\ 0 \\ -4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 4 \\ 5 \\ -3 \\ -6 \end{pmatrix}$$

Problem 2. Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$ be the set of polynomials of degree at most 2 with real coefficients. Here are three linear functionals on P_2:

$$\begin{aligned} \vec{\varphi}_1(\vec{f}) &= \vec{f}(-1) \\ \vec{\varphi}_2(\vec{f}) &= \vec{f}'(0) \\ \vec{\varphi}_3(\vec{f}) &= \vec{f}(3) \end{aligned}$$

Prove that $\vec{\varphi}_1$, $\vec{\varphi}_2$, and $\vec{\varphi}_3$ are a basis of P_2^* and find the basis of P_2 to which they are dual. (Hint: it is possible do both parts of this problem at the same time if you explain what you are doing!)

Problem 3. Let $\mathbb{C}$ be the set of complex numbers. In this problem, we will view $\mathbb{C}$ as a vector space over $\mathbb{R}$, with $\vec{0} = \overrightarrow{0 + i0}$, with $\overrightarrow{a + ib} + \overrightarrow{c + id} = \overrightarrow{(a + c) + i(b + d)}$, and with $e.(\overrightarrow{a + ib}) = \overrightarrow{(ea) + i(eb)}$ for all $a + ib, c + id \in \mathbb{C}$ and all $e \in \mathbb{R}$.

(i) Prove that $\mathbb{C}$ is an $\mathbb{R}$-vector space. (Hint: it is possible to do this quickly taking advantage of the known fact that $\mathbb{C}$ is a field containing $\mathbb{R}$.)

(ii) Let $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ be the "real part" function, $\mathrm{Re}(\overrightarrow{a + ib}) = a$ and let $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ be the "imaginary part" function, $\mathrm{Im}(\overrightarrow{a + ib}) = b$. Show that Re and Im are linear functionals.

(iii) Find a basis of $\mathbb{C}$ as a vector space over $\mathbb{R}$ with respect to which $\{\mathrm{Re}, \mathrm{Im}\}$ is the dual basis. **Justify your answer!**

(iv) What is the dimension of $\mathbb{C}$ as a vector space over $\mathbb{R}$? **Justify your answer!**

Problem 4. Suppose that V is a vector space over a field F and that $\{\vec{v}^1, \vec{v}^2, \vec{v}^3, \ldots\}$ is a basis of V. This means that the dimension of V is infinite!

(i) Show that, for each i, there is a unique linear functional $\vec{v}_i : V \to F$ such that

$$\vec{v}_i(\vec{v}^j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

(ii) Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$ is a linearly independent subset of V^*.

(iii) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$ is **not** a basis of V^*.