Graded Problem Set 3

Math 2130 — Fall 2022

due Friday, 2 December

1. Let A be the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$.

   (a) Compute the eigenvectors and eigenvalues of A.

   Solution. The characteristic polynomial is

   $$\det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (5-\lambda)(5-\lambda)-4 = \lambda^2-10\lambda+21 = (\lambda-7)(\lambda-3)$$

   so the eigenvalues are 7 and 3. Since $A-3I$ is rank 1, its columns are eigenvectors of A with eigenvalue 7. Therefore the 7-eigenspace of A is spanned by $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Similarly, the 3-eigenspace is spanned by $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$. We can divide by 2 to get simpler eigenvectors. □

   (b) Compute A^{2022}.

   Solution. Let $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and let $\Lambda = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$. Then $AX = X\Lambda$. So $A = X\Lambda X^{-1}$ and $A^{2022} = X\Lambda^{2022}X^{-1}$. The inverse of X is

   $$X^{-1} = \frac{1}{\det(X)} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} X.$$

Therefore

$$\begin{aligned} A^{2022} &= X\Lambda^{2022}X^{-1} = \frac{1}{2}X\Lambda^{2022}X \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 7^{2022} & 0 \\ 0 & 3^{2022} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 7^{2022} & 7^{2022} \\ 3^{2022} & -3^{2022} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 7^{2022}+3^{2022} & 7^{2022}-3^{2022} \\ 7^{2022}-3^{2022} & 7^{2022}+3^{2022} \end{pmatrix} \end{aligned}$$

□

(c) There are two possibilities for $\lim_{n\to\infty} \frac{A^n\vec{v}}{\|A^n\vec{v}\|}$ when $\vec{v}$ is *not* an eigenvector of A. (In other words, which direction can the vectors $A^n\vec{v}$ approach as n becomes large?)

Solution. We could write $\vec{v}$ as $a\vec{x}_1 + b\vec{x}_2$ where $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $A\vec{v} = a(7^n)\vec{x}_1 + b(3^n)\vec{x}_2$. Then

$$\lim_{n\to\infty} \frac{a(7^n)\vec{x}_1 + b(3^n)\vec{x}_2}{\|a(7^n)\vec{x}_1 + b(3^n)\vec{x}_2\|} = \lim_{n\to\infty} \frac{a\vec{x}_1 + b(3/7)^n\vec{x}_2}{\|a\vec{x}_1 + b(3/7)^n\vec{x}_2\|} = \frac{a\vec{x}_1}{\|a\vec{x}_1\|} = \pm\frac{\vec{x}_1}{\|\vec{x}_1\|}.$$

That is, depending on whether a is positive or negative, the limit either points in the direction of $\vec{x}_1$ or opposite it. □

2. Let $\vec{u} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and let $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$. Compute the eigenvectors and eigenvalues of the 3×3 matrix $A = \vec{u}\vec{v}^T$.

Solution. Since A has rank 1, its null space is 2-dimensional. It consists of the vectors perpendicular to $\vec{v}$, and these are spanned by $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The vector $\vec{u}$ is also an eigenvector since $A\vec{u} = (\vec{v}^T\vec{u})\vec{u}$. Its eigenvalue is $\vec{v}^T\vec{u} = 1(2) + 3(4) + 5(6) = 44$. □

3. Suppose that A and B are square matrices and that $\vec{x}$ is an eigenvector of A with eigenvalue 3 and an eigenvector of B with eigenvalue 5. Explain why $\vec{x}$ is also an eigenvector of $A^4 + ABA^2$ and compute its eigenvalue.

Solution.

$$(A^4 + ABA^2)\vec{x} = (3^4 + 3(5)(3^2)) = 27(8) = 216$$

□

4. Let W be the plane in $\mathbb{R}^3$ spanned by the vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. Let A be the matrix that reflects vectors across W.

(a) Find a nonzero vector $\vec{x}$ that is perpendicular to W.

Solution. $\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ □

(b) Find the eigenvalues and eigenvectors of A.

Solution. Since $\vec{u}$ and $\vec{v}$ are in W, they are fixed by the reflection, so they are eigenvectors with eigenvalue 1. Since $\vec{x}$ is perpendicular to W, it is an eigenvector with eigenvalue -1.

Let X be the eigenbasis matrix $X = \begin{pmatrix} \vec{u} & \vec{v} & \vec{x} \end{pmatrix}$. Let $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

We have $AX = X\Lambda$ so $A = X\Lambda X^{-1}$. □

(c) (Optional) Compute the matrix of A using the projection onto the plane W or using the projection onto the line $W^\perp$.

Solution. The projection on $W^\perp$ is given by the matrix $\frac{\vec{x}\vec{x}^T}{\vec{x}^T\vec{x}}$. The projection on W is therefore $I - \frac{\vec{x}\vec{x}^T}{\vec{x}^T\vec{x}}$. The reflection is the difference of projection on W and projection on $W^\perp$ so the formula is $I - 2\frac{\vec{x}\vec{x}^T}{\vec{x}^T\vec{x}}$. □

5. For which values of c is $A = \begin{pmatrix} 3 & 2 \\ c & 4 \end{pmatrix}$

(i) diagonalizable by a real change of coordinates;

(ii) diagonalizable by a non-real complex change of coordinates;

(iii) not diagonalizable?

Solution. The characteristic polynomial is $(3-\lambda)(4-\lambda)-2c = \lambda^2 - 7\lambda + 12 - 2c$. The discriminant of this polynomial is $49 - 4(12-2c) = 1 + 8c$. This is positive when $c > \frac{-1}{8}$, in which case we get two distinct real eigenvalues and therefore the matrix is diagonalizable by a real change of coordinates. If $c < \frac{-1}{8}$ then we get two distinct complex eigenvalues and the matrix is diagonalizable by a non-real complex change of coordinates. If $c = \frac{-1}{8}$ then the quadratic formula gives the repeated eigenvalue $\lambda = \frac{7}{2}$.

The $\frac{7}{2}$ eigenspace is

$$N\begin{pmatrix} 3-\frac{7}{2} & 2 \\ -\frac{1}{8} & 4-\frac{7}{2} \end{pmatrix} = N\begin{pmatrix} -\frac{1}{2} & 2 \\ -\frac{1}{8} & \frac{1}{2} \end{pmatrix}$$

since this matrix is rank 1, we will have a 1-dimensional eigenspace in this case, and the matrix will not be diagonalizable. □