Graded Problem Set 2

Math 2130 — Fall 2022

due Friday, October 7

1. Let

$$A = \begin{pmatrix} 2 & 17 & 13 & 71 & 77 \\ 11 & 6 & -16 & -47 & 50 \\ -5 & -22 & -12 & -75 & -105 \\ 2 & 4 & 0 & 6 & 25 \end{pmatrix}$$

   (a) Use http://math.colorado.edu/~jonathan.wise/math2130/ps2.html to compute the reduced row echelon form of A. In your answer, give the reduced row echelon form and the sequence of commands you used to find it. Please copy and paste the commands from your browser window (do not hand write them) to make it easier for me to check them.

   Solution.

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   It was obtained with this code:

```
A.r1/=2
A.r2 -= 11*A.r1
A.r3 += 5*A.r1
A.r4 -= 2*A.r1
A.r2 /= A[2,2]
A.r3 -= A.r2*A[3,2]
A.r4 += 13*A.r2
A.r3 /= A[3,5]
A.r4 -= A[4,5]*A.r3
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A.r2 -= A[2,5]*A.r3
A.r1 -= '77/2'*A.r3
A.r1 -= '17/2'*A.r2
```

□

(b) Find a matrix B such that $BA = \text{rref}(A)$. You may use the computer again for this part.

Solution.

$$B = \begin{pmatrix} 470 & 91 & 388 & 0 \\ 905 & 175 & 747 & 0 \\ -212 & -41 & -175 & 0 \\ 740 & 143 & 611 & 1 \end{pmatrix}$$

□

(c) Find a basis for the column space of A.

Solution.

$$\begin{pmatrix} 2 \\ 11 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 17 \\ 6 \\ -22 \\ 4 \end{pmatrix}, \begin{pmatrix} 77 \\ 50 \\ -105 \\ 25 \end{pmatrix}$$

□

(d) Find a basis for the null space of A.

Solution.

$$\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -5 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

□

(e) Find a matrix V such that $N(A) = C(V)$.

Solution.

$$\begin{pmatrix} 2 & 7 \\ -1 & -5 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

□

(f) Find a matrix U such that $C(A) = N(U)$.

Solution. We use the last row of B, since that corresponds to the only zero row of $\text{rref}(A)$:

$$U = \begin{pmatrix} 740 & 143 & 611 & 1 \end{pmatrix}$$

□

(g) Find the dimensions of the row space and left null space of A.

Solution. The row space has the same dimension as the column space: 3. We know that $\dim N(A^T) + \dim C(A^T) = 4$, the number of rows of A, so therefore the left null space must have dimension 1. □

(h) For which numbers b does $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ b \end{pmatrix}$ have a solution?

Solution. A vector $\vec{v}$ is in the column space of A if and only if $U\vec{v} = 0$, where U is the matrix from above. We compute $U \begin{pmatrix} 1 \\ 0 \\ 0 \\ b \end{pmatrix} = 740 + b$ so b must be -740. □

(i) Find the general solution to $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ b \end{pmatrix}$ when b satisfies your condition from the last part.

Solution. The solutions to $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ b \end{pmatrix}$ are the same as the solutions to $\mathrm{rref}(A)\vec{x} = B\begin{pmatrix}1\\0\\0\\b\end{pmatrix}$. This equation is:

$$\begin{pmatrix}1 & 0 & -2 & -7 & 0\\0 & 1 & 1 & 5 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{pmatrix}\vec{x} = \begin{pmatrix}470\\905\\-212\\0\end{pmatrix}$$

We get the particular solution $\vec{x}_p = \begin{pmatrix}470\\905\\0\\0\\-212\end{pmatrix}$. The general solution is

$$\vec{x} = \begin{pmatrix}470\\905\\0\\0\\-212\end{pmatrix} + x_3\begin{pmatrix}2\\-1\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}7\\-5\\0\\1\\0\end{pmatrix}$$

□

2. Remember that a subspace of $\mathbb{R}^n$ is a set V of vectors with the following properties:

   V1 $\vec{0}$ is in V,

   V2 if c is a scalar and $\vec{v}$ is in $\vec{V}$ then $c\vec{v}$ is in V.

   V3 if $\vec{v}$ and $\vec{w}$ is in V then $\vec{v} + \vec{w}$ is in V.

   None of the following examples is a subspace of $\mathbb{R}^2$. Explain why not in each case.

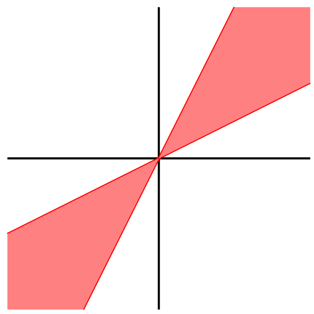
   (a) All vectors in $\mathbb{R}^2$ whose coordinates are integers.

   Solution. Fails V2. □

   (b) All solutions $\begin{pmatrix}x\\y\end{pmatrix}$ to the equation $x + 2y = 3$.

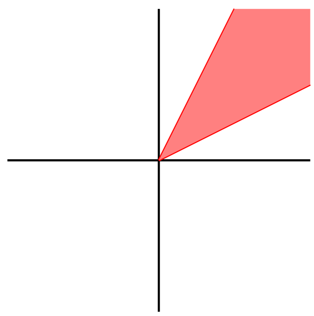
   Solution. Fails V1, V2, and V3. □

(c) The shaded area in red:



Solution. Fails V3. □

(d) The shaded area in red:



Solution. Fails V2. □

3. Find a 4×4 matrix A whose column space is spanned by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 7 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}$$

and whose null space is spanned by the vectors

$$\begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -2 \\ 5 \\ 1 \\ 1 \end{pmatrix}.$$

There are multiple correct approaches and multiple correct answers to this problem, so make sure to show your process clearly.

Solution. The null space contains the vectors

$$\begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 \\ 6 \\ 0 \\ 1 \end{pmatrix}.$$

These are the special solutions for the following RREF matrix

$$B = \begin{pmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 1 & -6 \end{pmatrix}$$

The matrix

$$C = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 2 \\ 7 & 2 \end{pmatrix}$$

has the correct column space and has null space $\{\vec{0}\}$. Since the column space of B is all of $\mathbb{R}^2$, the column space of CB is the same as the column space of C. Since C has null space $\{\vec{0}\}$, the null space of CB is the same as the null space of B. Therefore CB is the matrix we want. □

4. A 4×4 matrix A and a vector $\vec{b} \in \mathbb{R}^4$ are unknown. The set of solutions $\vec{x}$ to $A\vec{x} = \vec{b}$ contains a collection of 3 *linearly independent* vectors. What are the possibilities for $\dim N(A)$ and $\dim C(A)$?

   Solution. Let $\vec{x}_0, \vec{x}_1, \vec{x}_2$ be a linearly independent collection of 3 vectors that solve $A\vec{x} = \vec{b}$. Then $\vec{x}_1 - \vec{x}_0$ and $\vec{x}_2 - \vec{x}_0$ both solve $A\vec{x} = \vec{0}$. Furthermore, $\vec{x}_1 - \vec{x}_0$ and $\vec{x}_2 - \vec{x}_0$ are linearly independent. Therefore $\dim N(A) \geq 2$. The null space of A could therefore have dimension 2, 3, or 4. In these cases, the column space has dimensions 2, 1, and 0, respectively. □