Graded Problem Set 1

Math 2130 — Fall 2022

due Friday, September 2

Writeup: Do not consult any sources or notes from your investigation phase.

1. Suppose that A is an $m \times n$ matrix and that $\vec{x}$ and $\vec{y}$ are vectors such that $A\vec{x} = \vec{y}$. How many rows do $\vec{x}$ and $\vec{y}$ each have?

   Solution. $\vec{x}$ must have n rows and $\vec{y}$ must have m rows. □

2. Find a matrix A and vectors $\vec{x}$ and $\vec{b}$ that allow you to rewrite the system of linear equations

$$\begin{aligned} 8x + 4y + 2w &= 1 \\ y + z &= 0 \\ 2x + w &= -1 \end{aligned}$$

   as a vector equation $A\vec{x} = \vec{b}$. The entries of $\vec{x}$ should all be variables and the entires of A and $\vec{b}$ should all be real numbers.

   Solution.

$$A = \begin{pmatrix} 8 & 4 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

   □

3. Find all unit vectors perpendicular to $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

   Solution. $\frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $-\frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ □

4. Suppose that $\vec{v}$ and $\vec{w}$ are unknown vectors in $\mathbb{R}^n$ that make an angle of $\frac{\pi}{3}$. Assume that $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 6$. Compute $\|\vec{v} + \vec{w}\|$.

   Solution.

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2 &= (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) \\
&= \vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w} \\
&= \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\|\cos(\frac{\pi}{3}) + \|\vec{w}\|^2 \\
&= 25 + 2\cdot 5\cdot 6\cdot \frac{1}{2} + 36 \\
&= 91
\end{aligned}$$

   Therefore $\|\vec{v}+\vec{w}\| = \sqrt{91}$. □

5. Do the following sets of vectors generate a line, a plane, or all of $\mathbb{R}^3$? Give a very brief explanation of how you know in each case.

   (a) $\vec{u} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix}$

   Solution. This generates a line because the vectors are nonzero and the second vector is $\frac{3}{2}$ times the first. □

   (b) $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

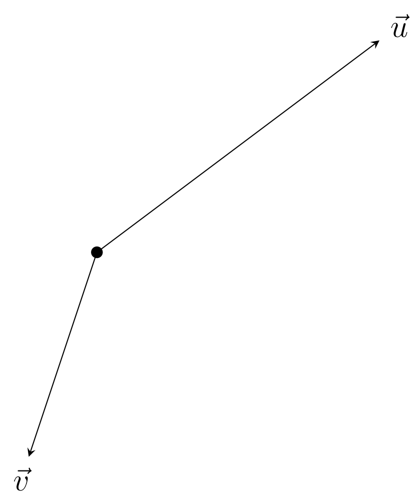
   Solution. The entries of the vectors add to zero so they do not generate all of $\mathbb{R}^3$ and they are not parallel so they do not generate a line. Therefore they must generate a plane. □

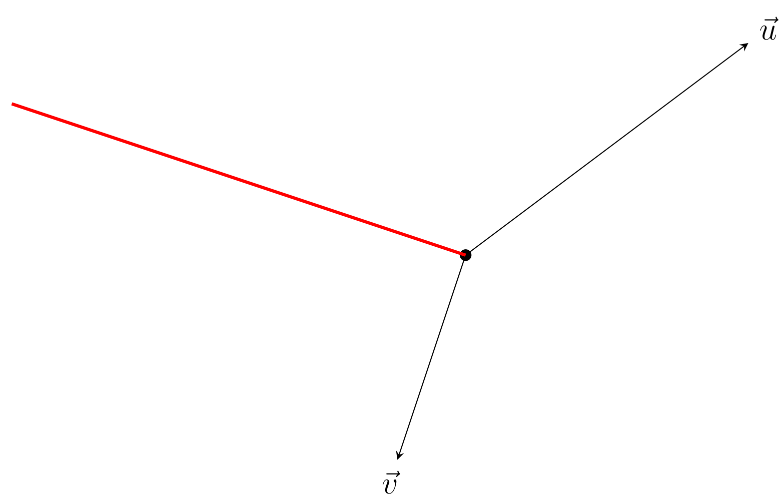
   (c) $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

   Solution. We know that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ generate all of $\mathbb{R}^3$. We can generate those from $\vec{u}$, $\vec{v}$, and $\vec{w}$ because $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$

$\frac{1}{2}(\vec{u}+\vec{v}-\vec{w})$ and $\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{2}(-\vec{u}+\vec{v}-\vec{w})$ and $\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{2}(-\vec{u}-\vec{v}+\vec{w})$.
Therefore $\vec{u}$, $\vec{v}$, and $\vec{w}$ generate all of $\mathbb{R}^3$. □

6. The following image shows 2 vectors, $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$. Shade the region containing all vectors $\vec{x}$ such that $\vec{x} \cdot \vec{u} < 0$ and $\vec{x} \cdot \vec{v} = 0$.





Solution. □

7. Suppose that $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{x}$ are all vectors in $\mathbb{R}^3$ and that we know the

following about their dot products:

$$\vec{u} \cdot \vec{v} = 0 \qquad \vec{u} \cdot \vec{x} = 0 \qquad \vec{v} \cdot \vec{x} = 0 \qquad \vec{w} \cdot \vec{x} = 0$$
$$\vec{u} \cdot \vec{u} = 1 \qquad \vec{v} \cdot \vec{v} = 1 \qquad \vec{w} \cdot \vec{w} = 1 \qquad \vec{x} \cdot \vec{x} = 1$$

(a) Is $\vec{x}$ a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$? Why or why not?

Solution. No: if $\vec{x} = a\vec{u}+b\vec{v}+c\vec{w}$ then $\vec{x}\cdot\vec{x} = a\vec{x}\cdot\vec{u}+b\vec{x}\cdot\vec{v}+c\vec{x}\cdot\vec{w} = 0$. But $\vec{x} \cdot \vec{x} = 1$.

□

(b) Is $\vec{w}$ a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{x}$? Why or why not?

Solution. Yes: since $\vec{v}$ is perpendicular to $\vec{u}$, the vector $\vec{u}$ and $\vec{v}$ do not lie on a line; since $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, it cannot lie in the same plane as $\vec{u}$ and $\vec{v}$. Therefore $\vec{u}$, $\vec{v}$, and $\vec{w}$ generate all of $\mathbb{R}^3$. In particular, $\vec{x}$ must be a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$. □

8. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are unknown vectors that all lie in a plane. Predict how many vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will solve the following equation:

$$x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$$

Your answer should either be that there are no solutions, there is exactly one solution, or there is more than one solution. Explain how you know.

Solution. There is more than one solution. There is certainly at least one solution: $x = y = z = 0$. Here is how we know there is another one: Two of $\vec{u}$ and $\vec{v}$ and $\vec{w}$ are already enough to generate the whole plane. If, for example, $\vec{u}$ and $\vec{v}$ generate the whole plane then $\vec{w} = a\vec{u} + b\vec{v}$ for some scalars a and b. But then $a\vec{u} + b\vec{v} - \vec{w} = \vec{0}$. So $x = a$, $y = b$, and $z = -1$ is another solution. □