Exam 3 Preview

Math 2130 — Fall 2022

2 November, 2022

1. The following matrix A has orthonormal columns but has some unknown entries, labelled with an asterisk ($*$). Assuming $\det(A) > 0$, write down as many entries of its cofactor matrix as you can.

$$A = \begin{pmatrix} \frac{1}{3} & * & \frac{2}{7} & * \\ -\frac{1}{4} & * & * & \frac{3}{8} \\ * & * & \frac{2}{5} & \frac{1}{4} \\ * & \frac{1}{2} & * & * \end{pmatrix}$$

Solution. The cofactor matrix is A. □

2. Suppose that A is a square matrix whose entries are all *integers*. For what values of $\det(A)$ does A have an invese whose entries are all integers? Explain your answer.

Solution. $\det(A) = \pm 1$ □

3. The following matrix has n rows and n columns, following the pattern begun below. Compute $\det(A_{100})$.

$$A_n = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ -1 & 1 & -1 & 0 & \cdots \\ 0 & -1 & 1 & -1 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Solution. $\det(A_{100}) = -1$ □

4. For what values of c is the following matrix invertible?

$$\begin{pmatrix} 16 & 12 & -12 \\ -13 & -6 & 11 \\ 3 & 6 & -1 \end{pmatrix} - cI$$

Solution. $c \neq 0, 4, 5$ □

5. In this problem $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}$ is an unknown 3×3 matrix with determinant 5. Let $\vec{x}$ be the following vector

$$\vec{x} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$$

and let $\vec{b} = A\vec{x}$. Compute the determinant of the matrix $\begin{pmatrix} \vec{b} & \vec{v}_3 & \vec{v}_1 \end{pmatrix}$.

Solution. $\det(A) = -20$ □

6. In this problem, A will be the 2×4 matrix shown below and B will be an unknown 4×2 matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}$$

What we know about B are the determinants of its 2×2 minors:

$$\det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \cancel{b_{31}} & \cancel{b_{32}} \\ \cancel{b_{41}} & \cancel{b_{42}} \end{pmatrix} = 0 \qquad \det \begin{pmatrix} b_{11} & b_{12} \\ \cancel{b_{21}} & \cancel{b_{22}} \\ b_{31} & b_{32} \\ \cancel{b_{41}} & \cancel{b_{42}} \end{pmatrix} = 3 \qquad \det \begin{pmatrix} b_{11} & b_{12} \\ \cancel{b_{21}} & \cancel{b_{22}} \\ \cancel{b_{31}} & \cancel{b_{32}} \\ b_{41} & b_{42} \end{pmatrix} = 0$$

$$\det \begin{pmatrix} \cancel{b_{11}} & \cancel{b_{12}} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ \cancel{b_{41}} & \cancel{b_{42}} \end{pmatrix} = -2 \qquad \det \begin{pmatrix} \cancel{b_{11}} & \cancel{b_{12}} \\ b_{21} & b_{22} \\ \cancel{b_{31}} & \cancel{b_{32}} \\ b_{41} & b_{42} \end{pmatrix} = 0 \qquad \det \begin{pmatrix} \cancel{b_{11}} & \cancel{b_{12}} \\ \cancel{b_{21}} & \cancel{b_{22}} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} = 5$$

Calculate the determinant of AB.

Solution. $\det(AB) = -18$ □

7. (Challenge problem) Find a relationship between the 2×2 minors of all 2×4 matrices.

Solution. $\det(A_{13}) \det(A_{24}) = \det(A_{12}) \det(A_{34}) + \det(A_{14}) \det(A_{23})$ □