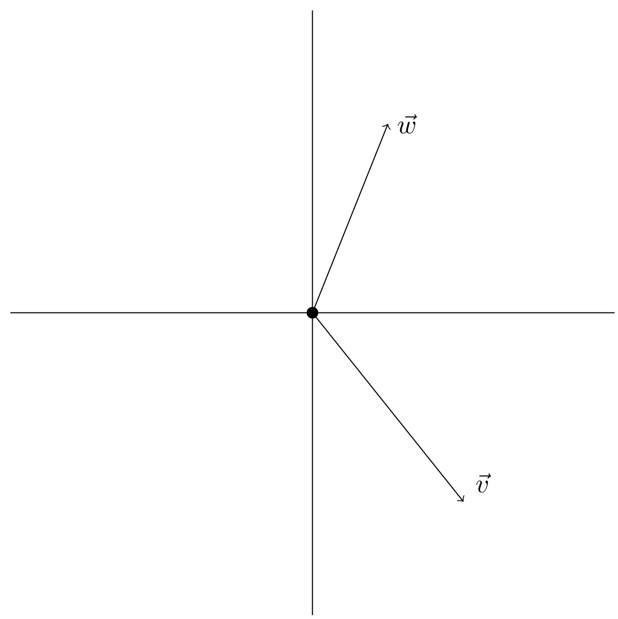
Handout 2: You could have invented determinants

Math 2130 — Fall 2022

24 October, 2022

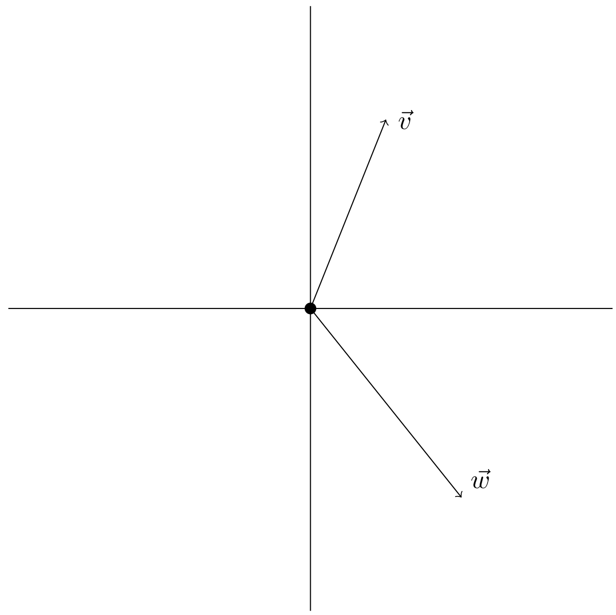
1. Draw the collection of all points $a\vec{v} + b\vec{w}$ where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.



2. What is the formula for the area of a parallelogram? (Say something about base and height.)
3. Label the picture with the terms from your formula.
4. Draw all points in the plane where you could move $\vec{w}$ (while holding $\vec{v}$ fixed) without changing the area of your parallelogram.
5. Find a vector $\vec{w}'$ on the positive y-axis such that the parallelogram spanned by $\vec{v}$ and $\vec{w}'$ has the same area as the parallelogram spanned by $\vec{v}$ and $\vec{w}$.
6. Now find a vector $\vec{v}'$ on the positive y-axis such that the parallelogram spanned by $\vec{v}'$ and $\vec{w}'$ has the same area as the original parallelogram.
7. Describe a process for finding the area of any parallelogram.
8. Carry out the above process for two particular vectors, for example $\vec{v} = (2, -2)$ and $\vec{w} = (1, 3)$. Does the process look familiar? If not, try writing $\vec{v}$ and $\vec{w}$ as the rows of a matrix at each step of the process.
9. Can you generalize this process to give meaning to the absolute value of the determinant of an $n \times n$ matrix?

Bonus questions

10. On the first page, we were able to move $\vec{v}$ to the positive x-axis and $\vec{w}$ to the positive y-axis by a series of steps in which each vector was always moving parallel to the other. What if we start with $\vec{v}$ and $\vec{w}$ switched? Can you get $\vec{v}$ to the positive x-axis and $\vec{w}$ to the positive y-axis now? Can you still calculate the area of the parallelogram? How can you tell in advance, by looking at $\vec{v}$ and $\vec{w}$, whether it will be possible to move $\vec{v}$ to the positive x-axis and $\vec{w}$ to the positive y-axis?



11. If you start with 3 independent vectors in $\mathbb{R}^3$, can you transform the first to the positive x-axis, the second to the positive y-axis, and the third to the positive z-axis by a sequence of moves where each vector always moves parallel to the other vectors? How far can you get before you run into trouble? Can you get all of the vectors on the right axes, even if they aren't all on the positive axes?

12. Make up an example of 3 independent vectors in $\mathbb{R}^3$ that can be transformed to the positive axes and another example that can't. Now, imagine yourself positioned inside the two parallelpipeds spanned by those two triples of vectors and look towards the origin. What is different about what you see? It might help to imagine slicing the parallelpiped by a plane near the origin, so that the slice looks like a triangle inside the plane. What is different about the triangles?

13. Make a conjecture about the meaning of the determinant in 3 dimensions. Can you explain both the absolute value and the sign of the determinant? Can you generalize your interpretation to n dimensions?

14. Suppose you had four vectors in $\mathbb{R}^4$ and positioned yourself inside the parallelpiped that they span, looking towards the origin. Slice the parallelpiped by a hyperplane near the origin. What sort of shape does it make inside of that 3-dimensional hyperplane?

Determinant and projections

15. Suppose that $\vec{v} = (2, -2)$ and $\vec{w} = (1, 3)$. Find the projection $\vec{y}$ of $\vec{w}$ onto the subspace perpendicular to $\vec{v}$. Compare $\|y\|\,\|v\|$ and $\det \begin{pmatrix} \vec{v}^T \\ \vec{w}^T \end{pmatrix}$. What happens if $\vec{v}$ and $\vec{w}$ are switched? Can you explain the relationships you observe? Can you generalize them to arbitrary vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$?

16. What is the relationship between $\det \begin{pmatrix} \vec{v}^T \\ \vec{w}^T \end{pmatrix}$ and $\det \begin{pmatrix} \vec{w}^T \\ \vec{v}^T \end{pmatrix}$? How does the sign of the determinant relate to the angle from $\vec{v}$ to $\vec{w}$?

17. Can you describe another formula for the determinant of a 2×2 matrix, involving the length of a projection and the sign of the angle from $\vec{v}$ to $\vec{w}$?

18. Consider the matrix
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
Find the projections $\vec{w}_2$ and $\vec{w}_3$ of the second and third rows onto the plane perpendicular to the first row.