Exam 2 Redux

Math 2130 — Fall 2022

19 October, 2022

Instructions: Work alone, using only paper and a writing implement. Make sure your name is on every page. Some of the problems below may have more than one correct solution, and some solutions may be more efficient than others, so spend some time considering the possible approaches before starting to calculate. Remember that your grade will be based on the understanding you demonstrate, not just the correctness of your answer, so be sure to include comprehensible justification for your work.

1. Let V be the hyperplane in $\mathbb{R}^5$ with the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 0$.

   (a) Write the vector

   $$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

   as the sum of a vector in V and a vector in $V^\perp$. (Hint: we have discussed many ways to do this. Some approaches may be quicker than others.)

   (b) Describe some ways you can check your answer.

2. Find an equation whose solutions are the coefficients a and b in the equation $y = ax + b$ of the line that best fits the points $P_1 = (0,0)$, $P_2 = (1,1)$, $P_3 = (2,4)$, and $P_4 = (3,9)$. The meaning of 'best' here is that a and b should minimize the length of the following vector:

$$\begin{pmatrix} ax(P_1) + b - y(P_1) \\ ax(P_2) + b - y(P_2) \\ ax(P_3) + b - y(P_3) \\ ax(P_4) + b - y(P_4) \end{pmatrix}$$

Compute the total error of your line.

$$C = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} \square \\ \square \end{pmatrix}$$

3. Suppose that V and W are subspaces of $\mathbb{R}^n$ and $V \cap W$ is *not* the zero subspace. Let P be the projection matrix onto V and let Q be the projection matrix onto W. Mark each of the following statements as *always*, *sometimes*, or *never* true. Explain your answers.

   (a) $P + Q$ is the projection matrix for $V + W$. (Recall that $V + W$ consists of all vectors that can be written as a sum of a vector in V and a vector in W.)

   (b) PQ is the projection matrix for $V \cap W$. (Recall that $V \cap W$ is the subspace consisting of all vectors that are in both V and W.)

   (c) $I - P$ is the projection matrix for $V^\perp$. (Recall that $V^\perp$ consists of all vectors that are orthogonal to V.)

4. The matrix B has orthogonal (not orthonormal) columns. Find a left inverse B' of B and a matrix C such that $A = BC$.

$$A = \begin{pmatrix} 16 & -2 & -2 \\ -16 & -22 & -8 \\ -24 & -24 & 18 \\ 8 & 11 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -3 & -1 \\ -2 & -4 & 2 \\ 1 & 1 & 7 \\ 0 & 1 & -2 \end{pmatrix}$$

$$B' = \begin{pmatrix} \Box & \Box & \Box & \Box \\ \Box & \Box & \Box & \Box \\ \Box & \Box & \Box & \Box \end{pmatrix}$$

$$C = \begin{pmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{pmatrix}$$

5. Consider the following vectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Find orthogonal vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ such that

$$\operatorname{span}(\vec{w}_1) = \operatorname{span}(\vec{v}_1) \qquad \operatorname{span}(\vec{w}_1, \vec{w}_2) = \operatorname{span}(\vec{v}_1, \vec{v}_2) \qquad \operatorname{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3).$$

6. Suppose that P and Q are matrices with orthonormal columns. Answer the following questions with justification.

   (a) What are the largest and smallest possible entries of Q?

   (b) Does PQ have orthonormal columns? Answer either *always*, *sometimes*, or *never*.

   (c) Does Q^T has orthonormal columns? Answer either *always*, *sometimes*, or *never*. Does your answer change if Q is square?