Name: ____________________

Final Exam

Math 2130-003 — Fall 2022

11 December 2022

Instructions: Work alone, using only paper and a writing implement. Make sure your name is on every page. Remember that your grade will be based on the understanding you demonstrate, not just the correctness of your answer, so be sure to include comprehensible justification for your work. Most questions do not require lengthy calculations, so consider your approach to each problem carefully.

1. For which values of c is the following matrix invertible?

$$\begin{pmatrix} 3 & 2 & c \\ 1 & 0 & 1 \\ c & 5 & 1 \end{pmatrix}$$

Solution. By expansion along the middle row, the determinant is $-1(2-5c) - 1(15-2c) = 7c - 17$. Therefore the matrix is invertible as long as $c \neq \frac{17}{7}$. □

2. There is an unknown 4×5 matrix $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{pmatrix}$ (each of the vectors $\vec{v}_i$ has 4 rows). Its reduced row echelon form is shown below:

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 6 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer the following questions about A. Some of the answers may need to be phrased in terms of the vectors $\vec{v}_1, \ldots, \vec{v}_5$.

(a) Give a basis for $C(A)$.

Solution. $\vec{v}_1, \vec{v}_3$ □

(b) Give a basis for $N(A)$.

Solution. $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -8 \\ 0 \\ 1 \end{pmatrix}$ □

(c) Find the general solution to the following equation: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{v}_5$.

Solution.

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -6 \\ 0 \\ -5 \\ 1 \end{pmatrix}$$

□

(d) Let $A' = \begin{pmatrix} \vec{v}_3 & \vec{v}_5 \end{pmatrix}$. What is $\dim C(A')$? What is $\dim N(A')$? Explain.

Solution. $\dim C(A') = 1$ and $\dim N(A') = 1$. □

(e) There is an $m \times n$ matrix B and a $p \times q$ matrix C such that BAC is invertible. What are the possibilities for the values of m, n, p, and q?

Solution. $n = 4$ and $p = 5$. Since BAC can have at most 2 independent columns, its rank is ≤ 2. Since it is invertible, it must be square and the rank must be the same as the number of columns. Therefore $m = q = 2$ or $m = q = 1$ or $m = q = 0$. □

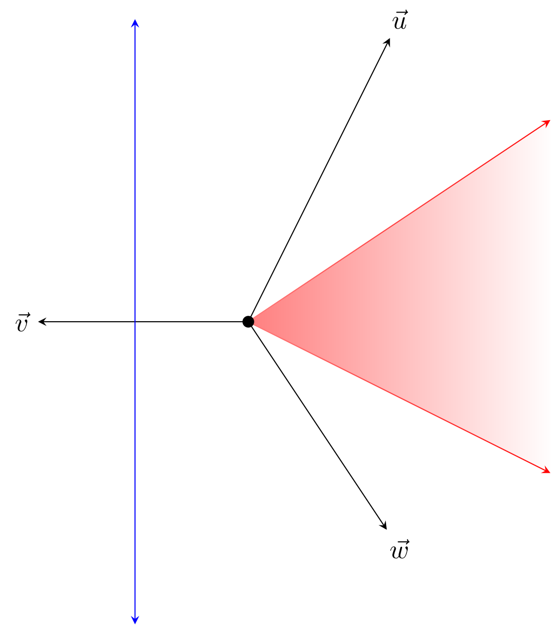
3. In the image below, indicate the following two regions:

   (a) the region containing all vectors $\vec{x}$ such that $\vec{u}^T\vec{x} > 0$ and $\vec{w}^T\vec{x} > 0$;

   (b) the region containing all vectors $\vec{y}$ such that $|\vec{v}^T\vec{y}| = \frac{1}{2}\|\vec{v}\|^2$.

   Label your drawing clearly.

   Solution. First answer in red, second in blue.



□

4. There are five unknown vectors $\vec{v}_1, \dots, \vec{v}_5$ in $\mathbf{R}^4$ with the following properties:

   (i) $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_5$ are linearly indepedent.

   (ii) $3\vec{v}_1 + 2\vec{v}_3 = \vec{0}$

   (iii) $\vec{v}_1 + \vec{v}_2 + \vec{v}_4 + \vec{v}_5 = \vec{0}$

   Find the reduced row echelon form of the matrix $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{pmatrix}$.

   Solution.

$$\begin{pmatrix} 1 & 0 & -\frac{3}{2} & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

□

Name:

5. Compute the reduced row echelon form of the following matrix (Hint: multiplying the matrices shouldn't be the first step):

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \\ 11 & 12 \\ 13 & 14 \\ 15 & 16 \\ 17 & 18 \\ 19 & 20 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

Solution. We can do row operations on the matrix on the left without changing the RREF of the product. Since the columns of the matrix on the left are independent, we can transform it to this one:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

The product is then this matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & 9 \\ 0 & 1 & 2 & \cdots & 8 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We can put it in RREF by subtracting twice the second row from the first. Here is what we get:

$$\begin{pmatrix} 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

□

6. Let $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Let $V = C(A)$ where A is the matrix below:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Find a basis for $V^\perp$.
Solution.

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

□

(b) Write $C(A)$ as the nullspace of a matrix B.

Solution.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

□

(c) Write $\vec{b}$ as the sum of a vector in V and a vector in $V^\perp$.

Solution. The projection on V is $A(A^TA)^{-1}A^T\vec{b}$. We have $A^TA = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A^T\vec{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$.

Therefore $A(A^TA)^{-1}A^T\vec{b} = \frac{1}{2}AA^T\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 3 \end{pmatrix}$. Therefore $\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$. □

7. Let $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find an orthonormal basis $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ of $\mathbb{R}^3$ including a multiple of $\vec{v}$. There are many correct answers to this question.

Solution. We take $\vec{x}_1 = \frac{1}{\sqrt{3}}\vec{v}$. The vector $\vec{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is orthogonal to $\vec{x}_1$ and has length 1.

The vector $\vec{x}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is orthogonal to both of these. □

(b) Create a matrix X whose columns are the vectors of the basis you found in the last part. What is the inverse of X?

Solution.

$$X = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$X^{-1} = X^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

□

(c) Find a matrix that rotates $\mathbb{R}^3$ around $\vec{v}$ by $\frac{\pi}{2} = 90°$. A rotation in either direction (clockwise or counterclockwise) is acceptable. Your answer can be written as a product of matrices, as long as those matrices' entries are explicit numbers.

Solution. We have $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = \vec{v}_3$ and $A\vec{v}_3 = -\vec{v}_2$, so

$$AX = X\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Therefore:

$$A = X\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}X^{-1}$$

□

8. Suppose that A is a matrix whose *columns* all sum to 3. Find at least one real (right) eigenvalue of A. Make sure to justify your answer. Be careful about the difference between left eigenvectors and right eigenvectors.

   Solution. Let $\vec{v}$ be the vector of all 1s. Then $\vec{v}^T A = 3\vec{v}^T$. Therefore 3 is a left eigenvalue of A. But the left and right eigenvalues of A are the same. □

9. Suppose that A is an orthogonal matrix (this means that the columns are orthonormal).

   (a) Suppose that $\vec{x}$ is a nonzero eigenvector of A. What are the possible values of $\frac{\|A\vec{x}\|}{\|\vec{x}\|}$?

   Solution. We get $\|A\vec{x}\|^2 = \vec{x}^T A^T A\vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2$. Therefore $\|A\vec{x}\| = \|\vec{x}\|$. □

   (b) What are the possible eigenvalues of A?

   Solution. If $A\vec{x} = \lambda\vec{x}$ has the same length as $\vec{x}$ then $\lambda = \pm 1$. □

10. Compute e^A for *one* of the following two values of A. **You don't have to do both parts!** Your answer can be a product of matrices as long as those matrices' entries are explicit numbers.

    (a) $A = \begin{pmatrix} 0 & -3 \\ 4 & 7 \end{pmatrix}$

    Solution. The characteristic polynomial is $\lambda^2 - 7\lambda + 12 = 0$, so the eigenvalues are $\lambda = 3, 4$. The eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$, respectively. Let $X = \begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix}$. The inverse of X is $X^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}$. Therefore $A = X\Lambda X^{-1}$ where $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and

    $$\begin{aligned} e^A = Xe^\Lambda X^{-1} &= \frac{1}{7}\begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^3 & 0 \\ 0 & e^4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 4e^3 & 3e^3 \\ -e^4 & e^4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4e^3 + 3e^4 & -3e^3 - 3e^4 \\ 4e^3 - 4e^4 & 3e^3 + 4e^4 \end{pmatrix} \end{aligned}$$

    □

    (b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (Hint: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.)

    Solution. We have

    $$e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} = I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

    Therefore

    $$\begin{aligned} e^{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} &= e^{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \\ &= \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e & \frac{e}{2} \\ 0 & e & e \\ 0 & 0 & e \end{pmatrix} \end{aligned}$$

    □